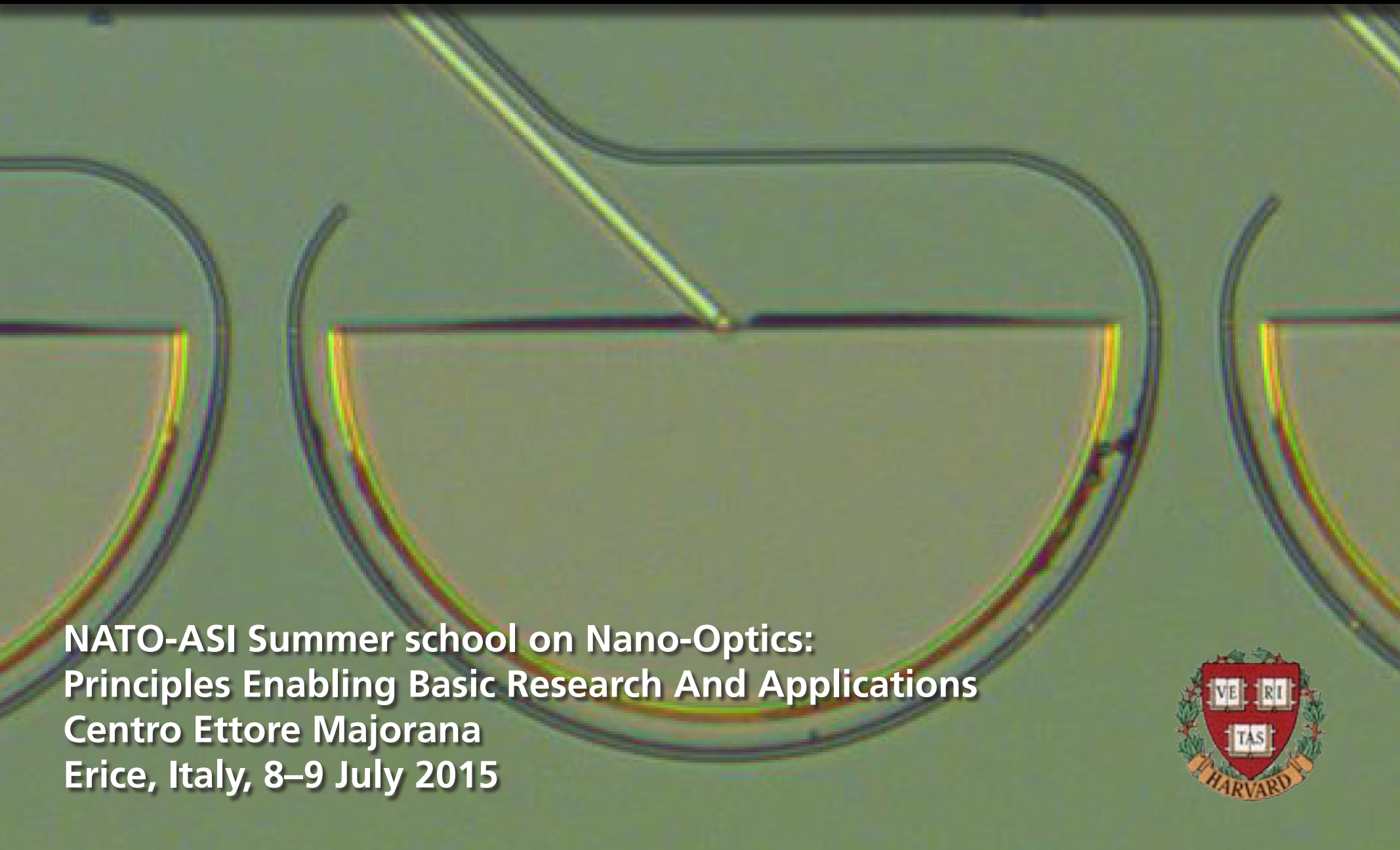
Manipulating Light at the Nanoscale



NATO-ASI Summer school on Nano-Optics:
Principles Enabling Basic Research And Applications
Centro Ettore Majorana
Erice, Italy, 8–9 July 2015

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@eric__mazur

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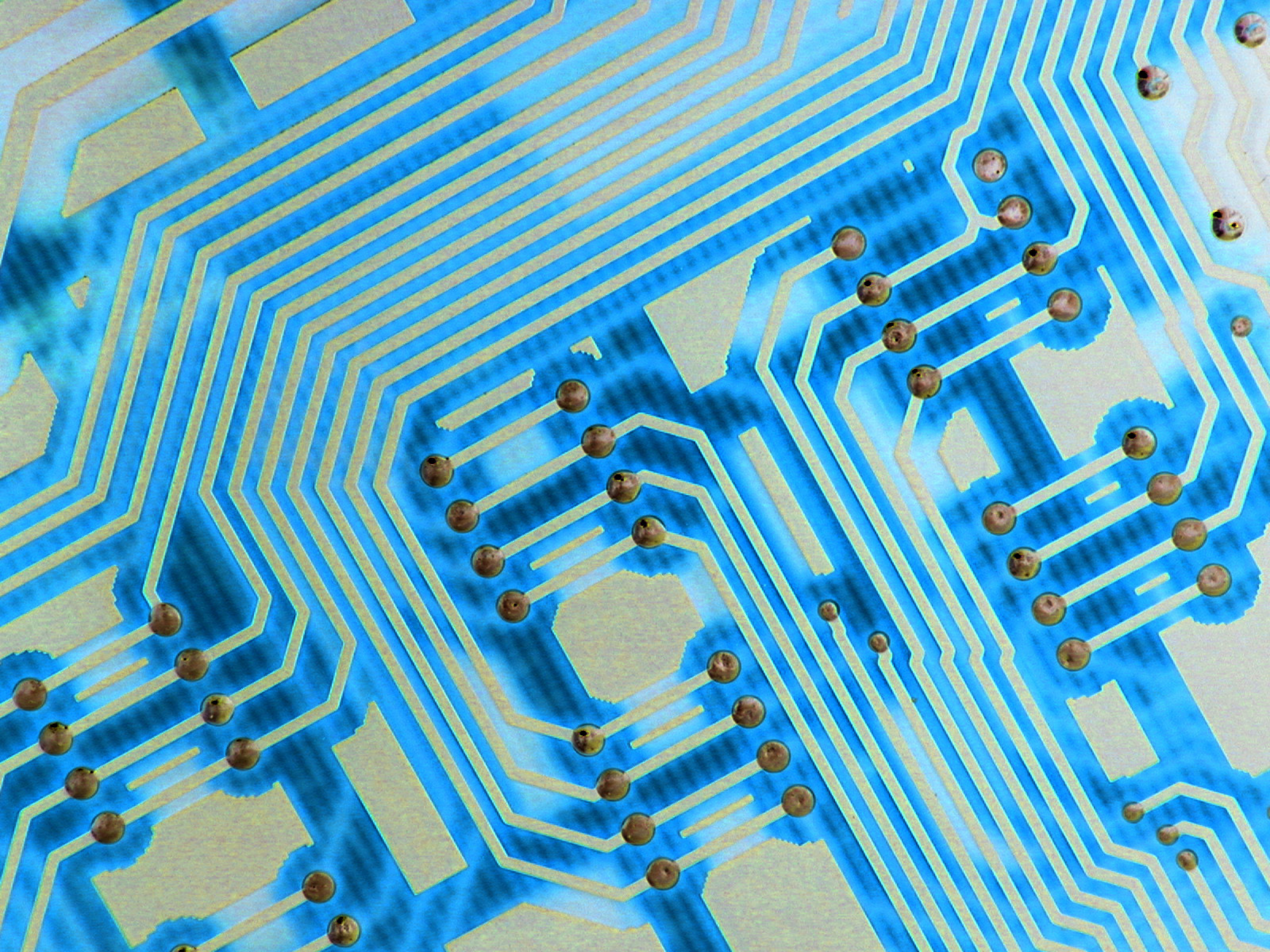

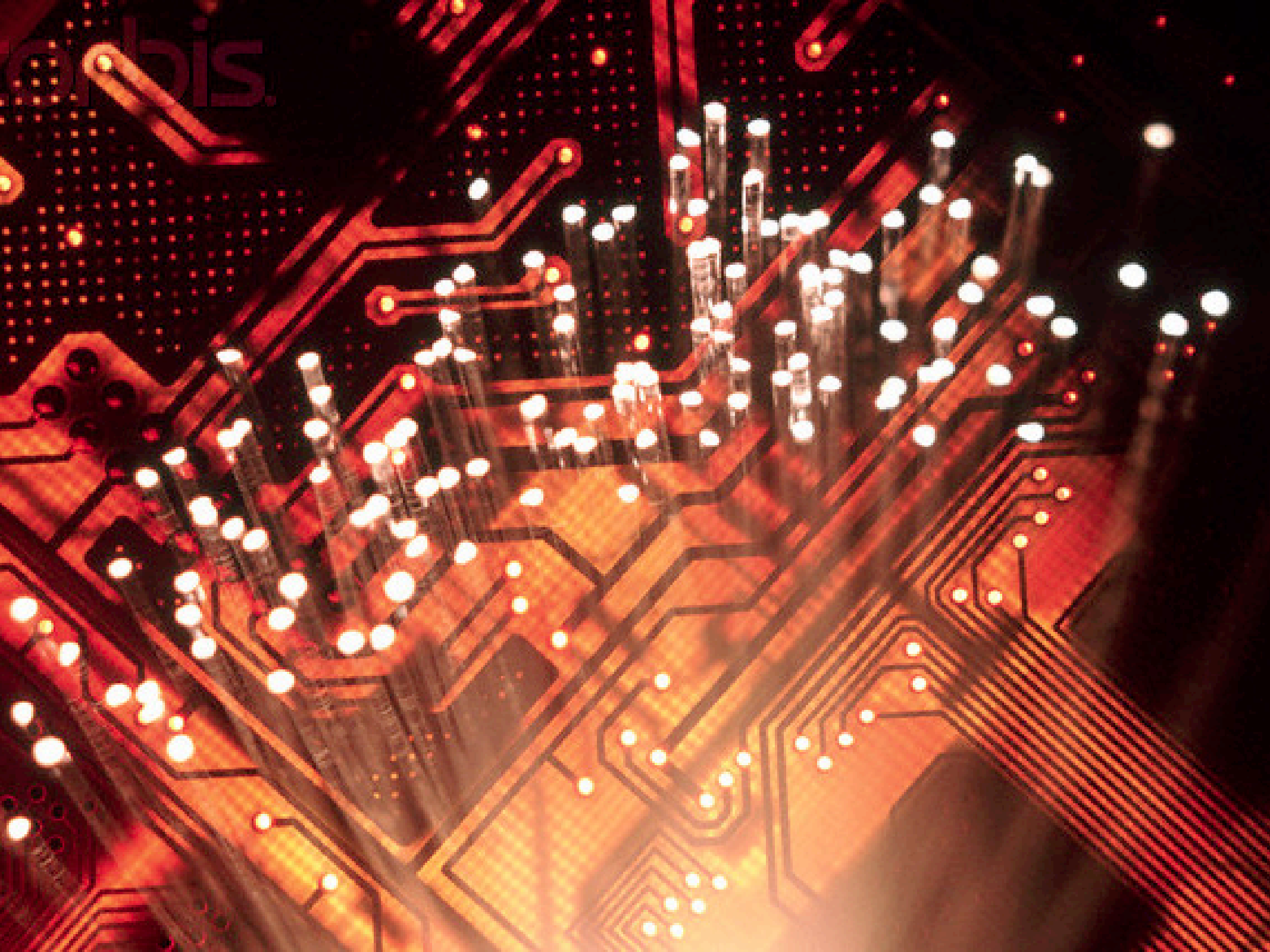

Outline

- optical properties of materials
- dispersion of pulses
- nonlinear optics
- waveguiding
- engineering the index

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In dispersive media $n = n\,(\omega)$ **.**

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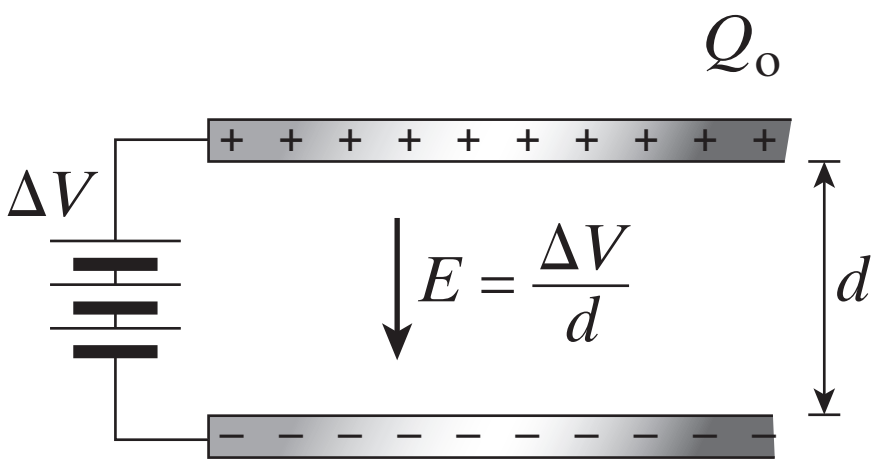
Dielectric constant measures increase in capacitance

$$\epsilon = \frac{C_d}{C_o}$$



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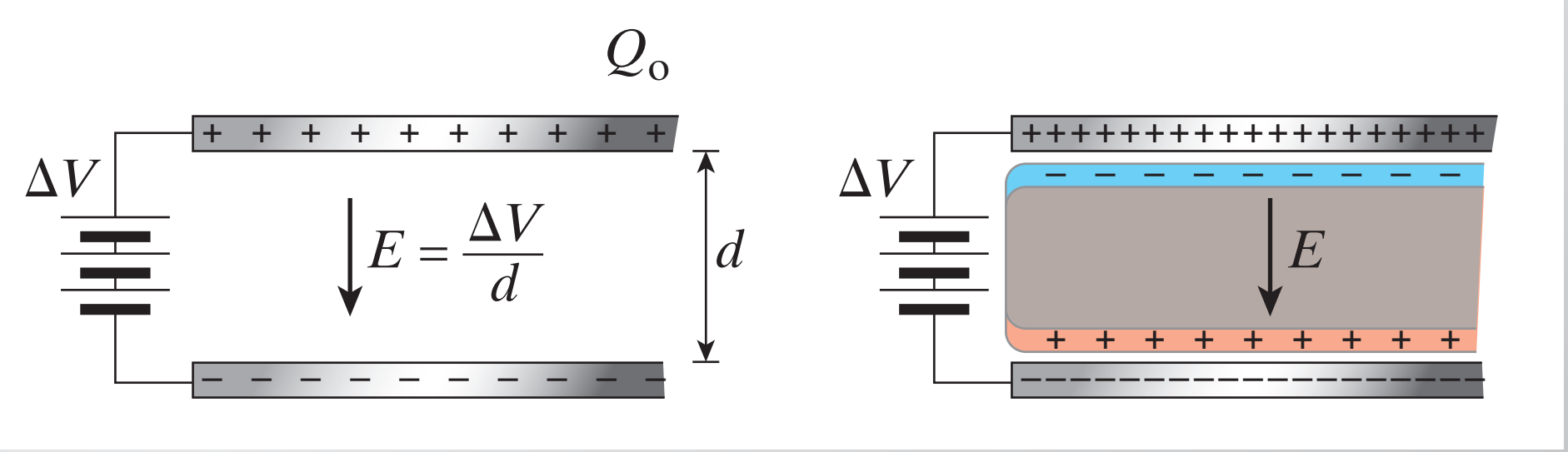

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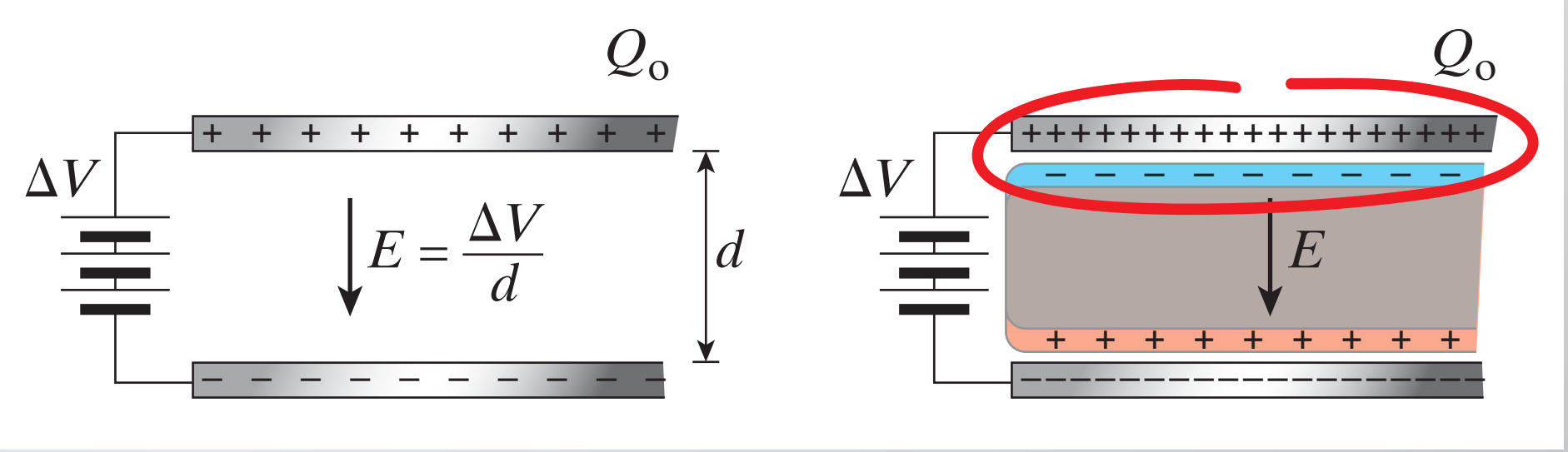

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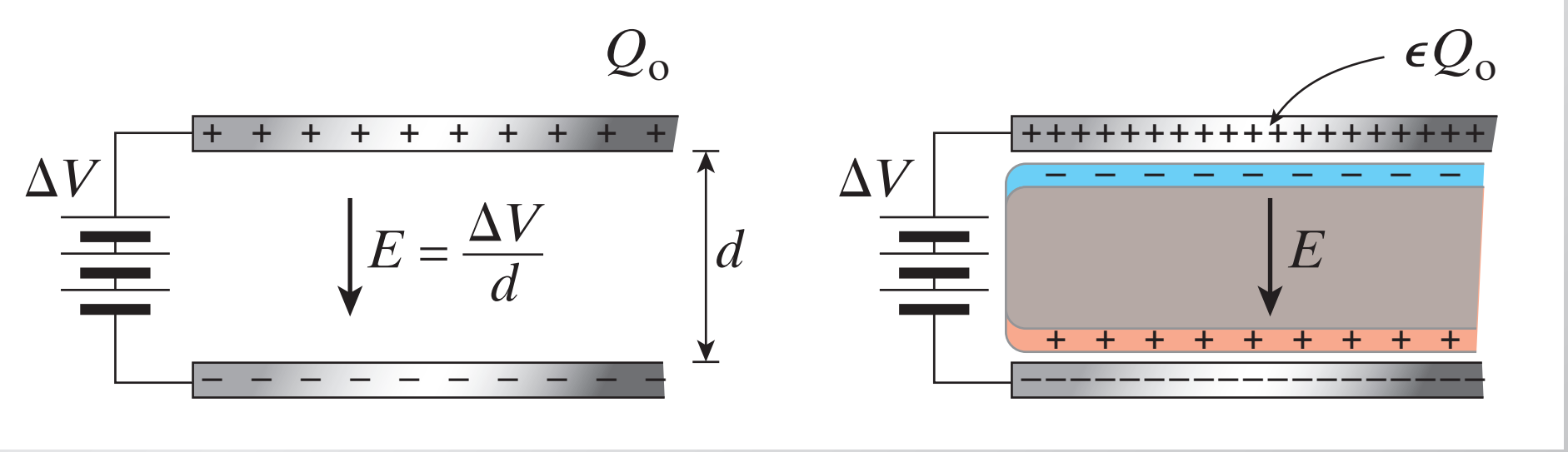
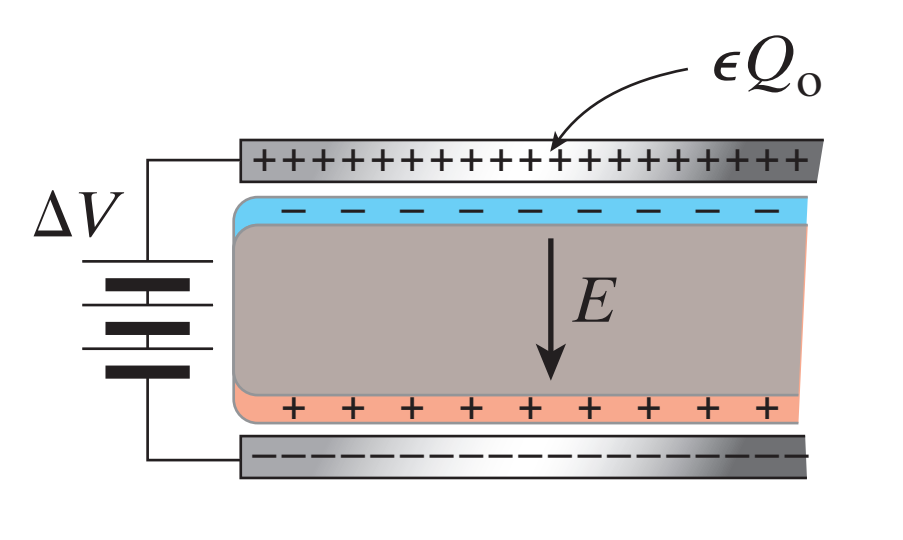


Propagation of EM wave through medium

Electric field is sum of two contributions

$$E = E_{free} + E_{bound} = \varepsilon E - P$$



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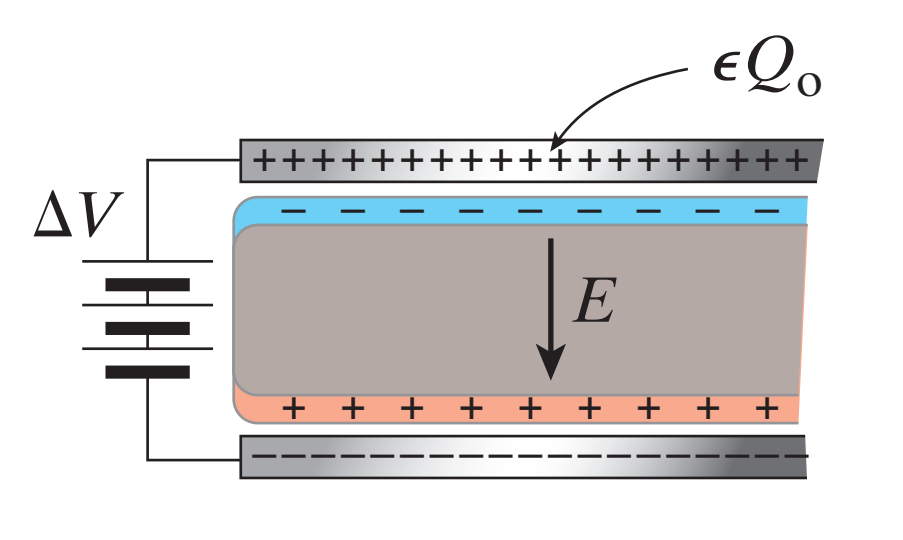

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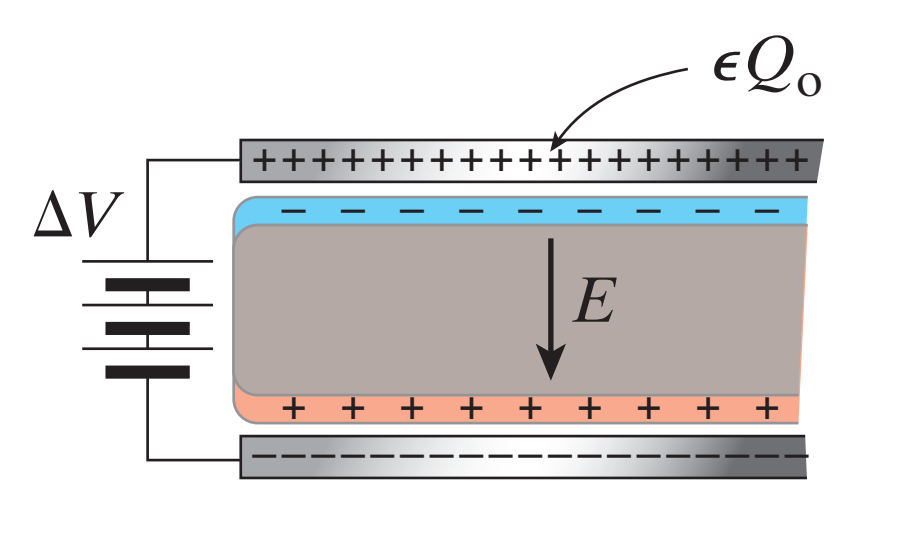

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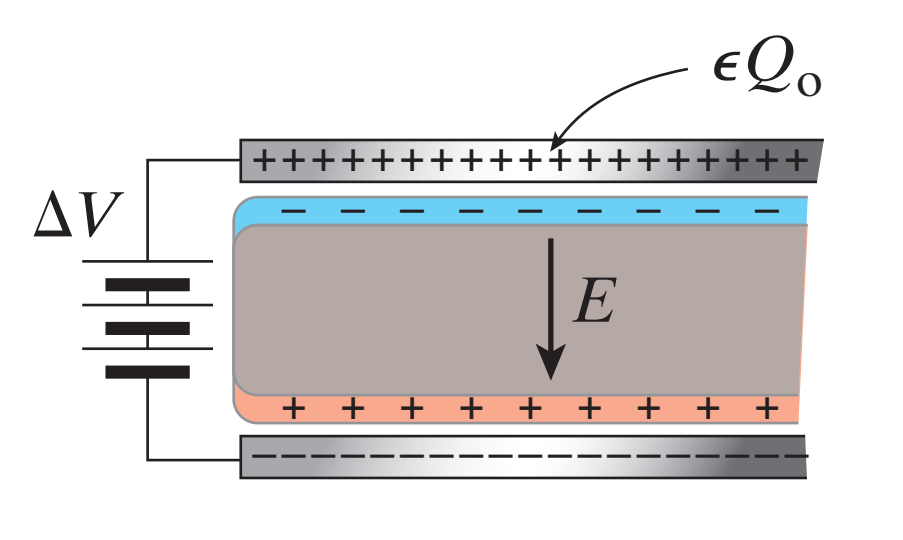


so

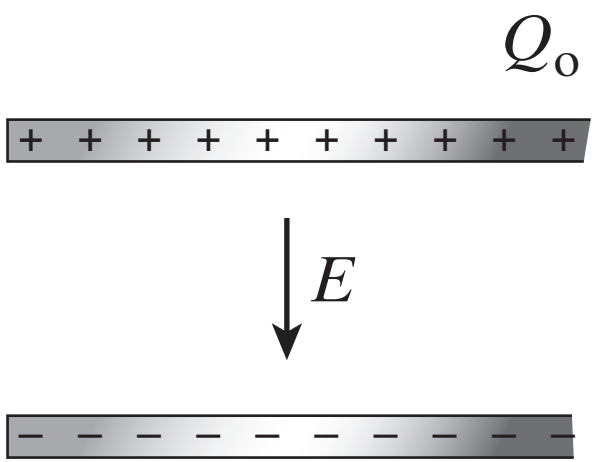
$$E = \varepsilon E - \chi E$$

or

$$\varepsilon = 1 + \chi$$

Propagation of EM wave through medium

Alternatively ϵ is measure of attentuation of electric field



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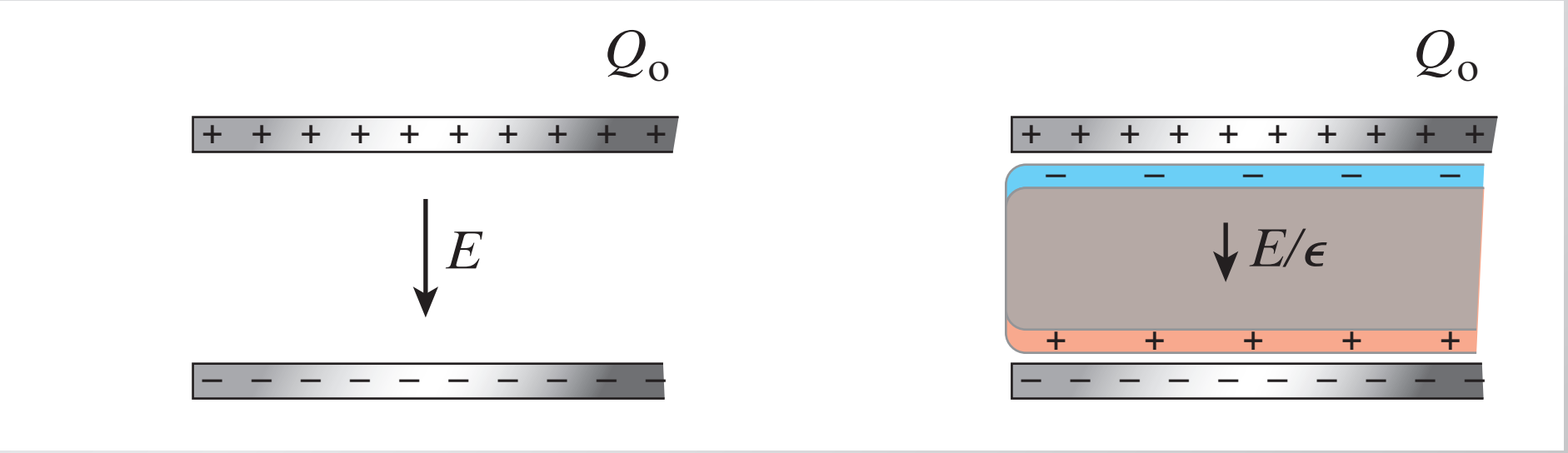
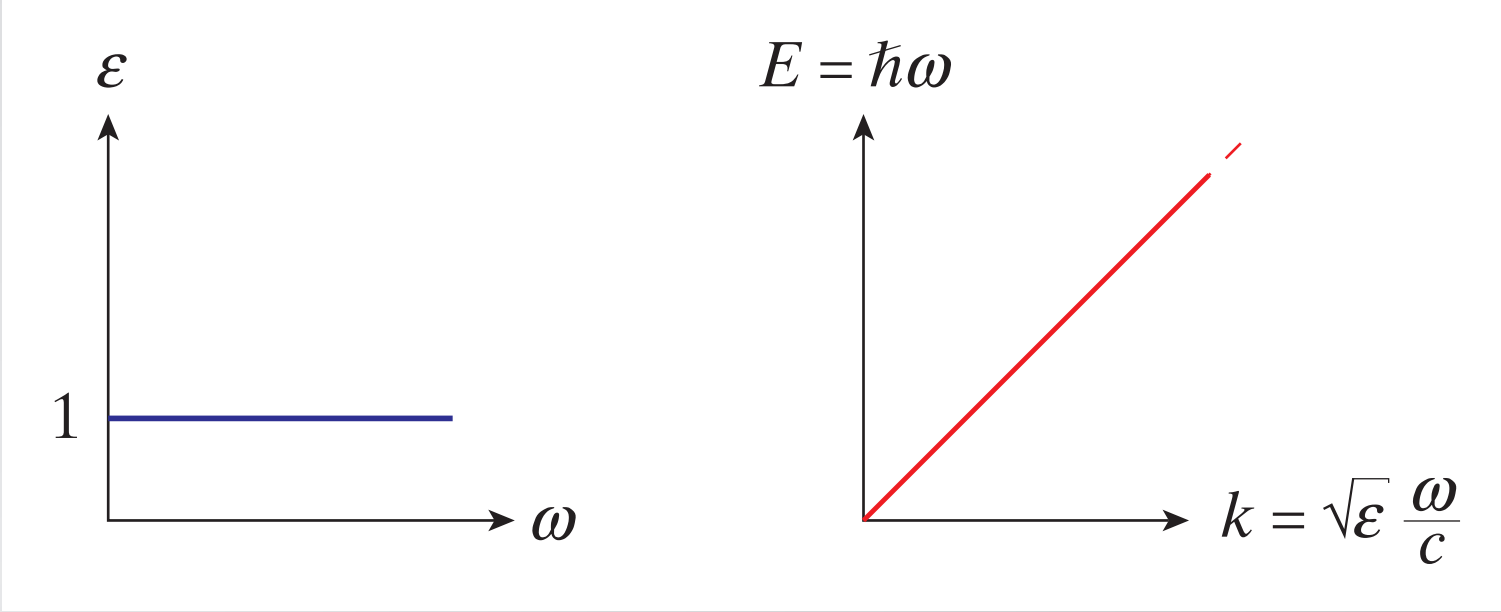


Propagation of EM wave through medium

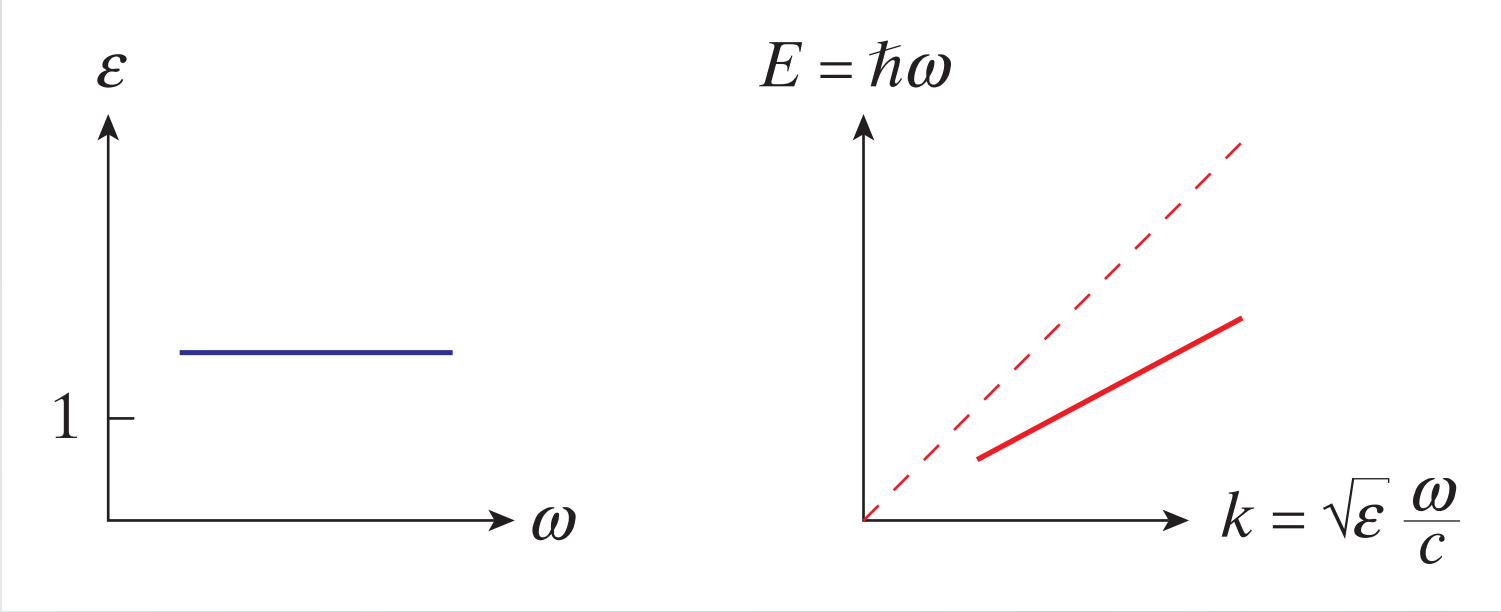
In vacuum: $$f\lambda = \frac{\omega}{k} = c \qquad \Rightarrow \qquad \omega = c\,k$$



Propagation of EM wave through medium

In medium:

$$v = \frac{c}{\sqrt{\epsilon(\omega)}} = \frac{c}{n} \qquad \Rightarrow \qquad \omega = \frac{c}{\sqrt{\epsilon}} k$$



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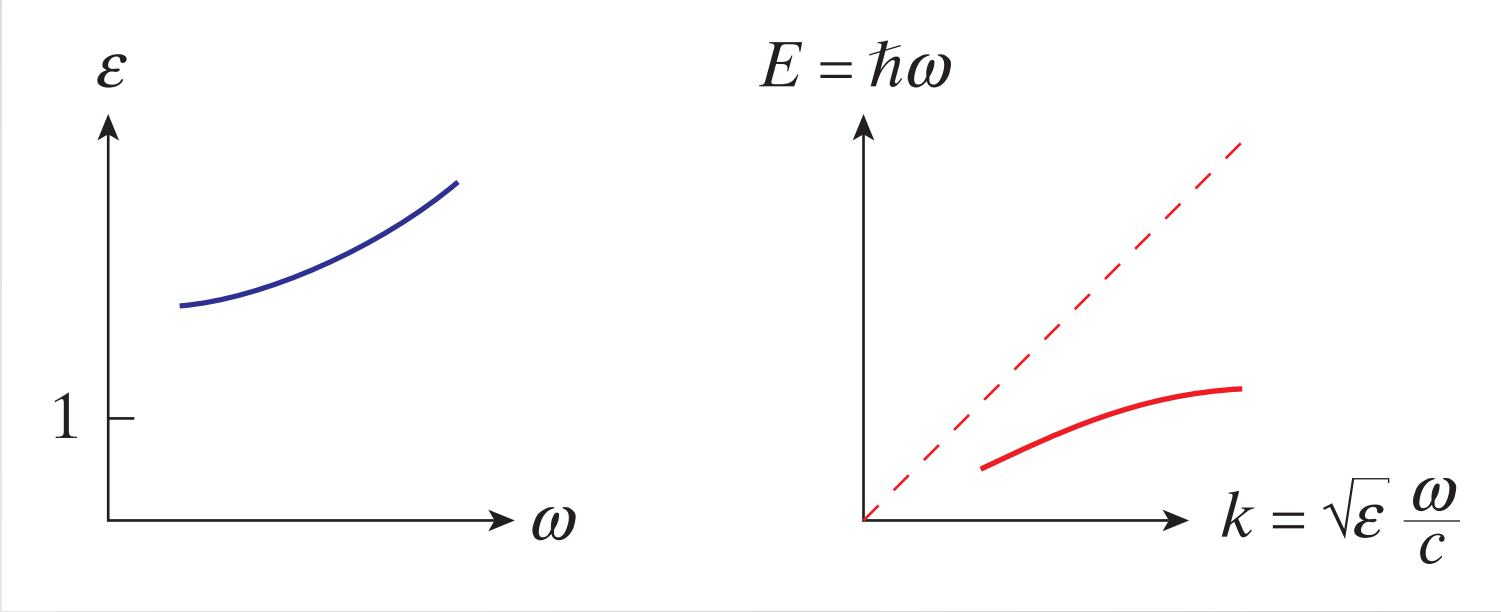
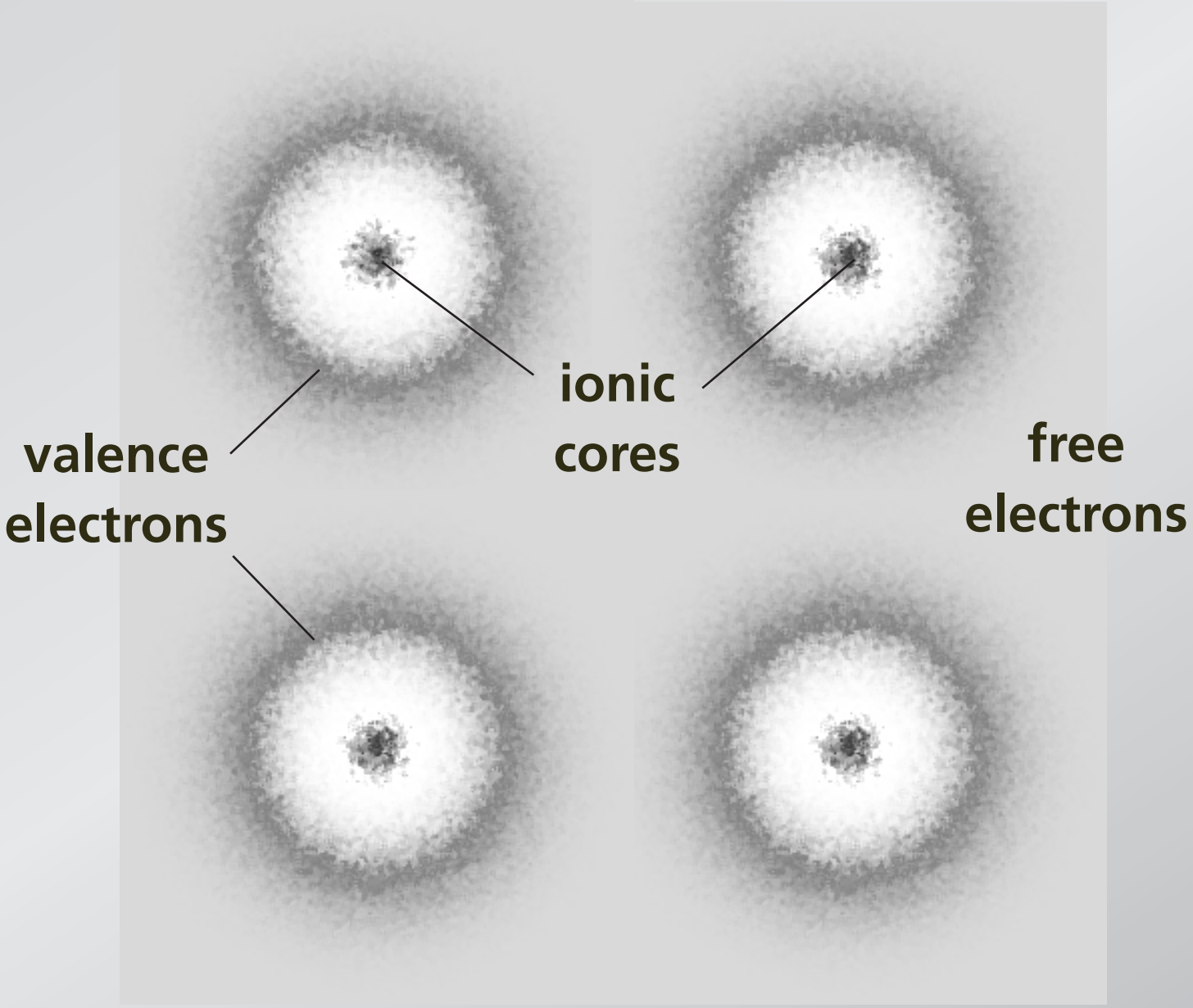
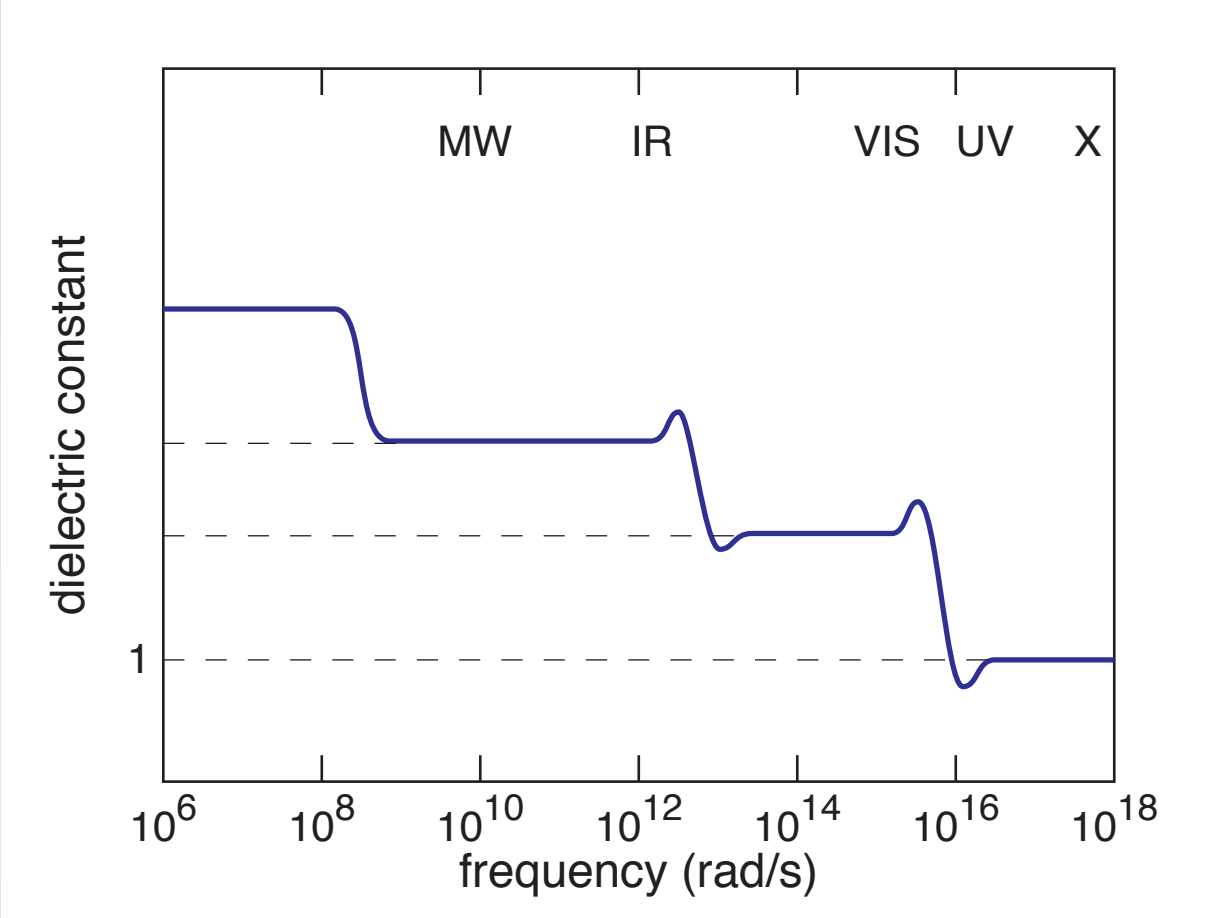


Which charges participate?



Dielectric function



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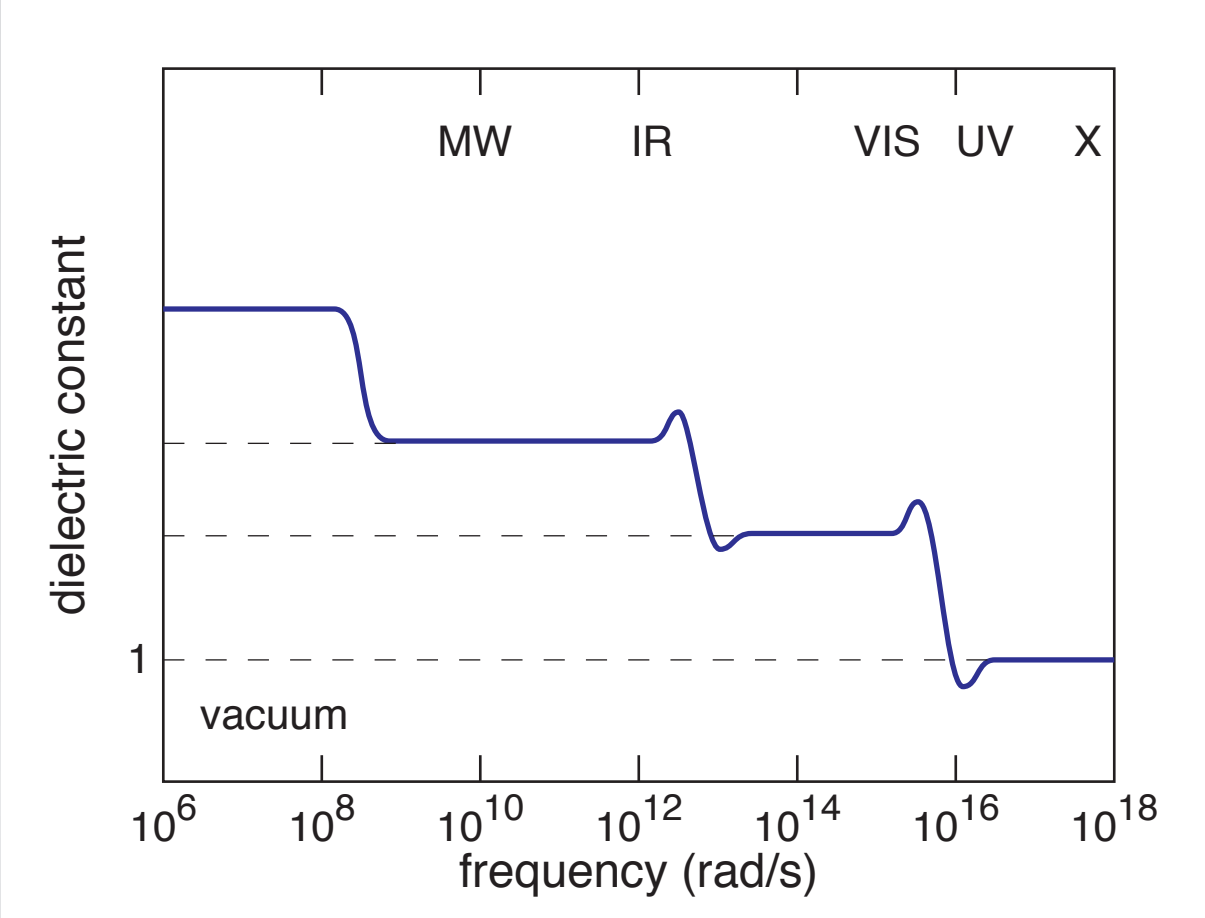

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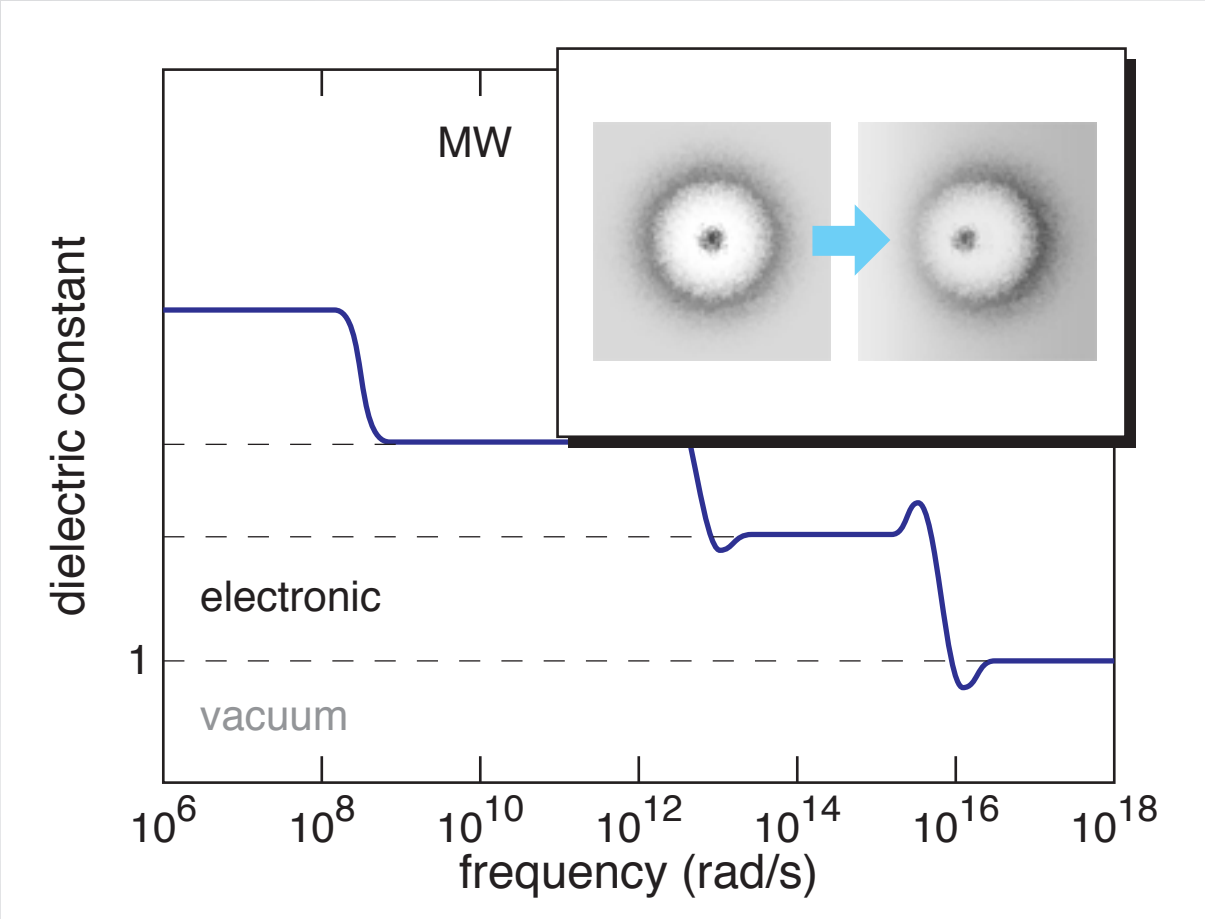

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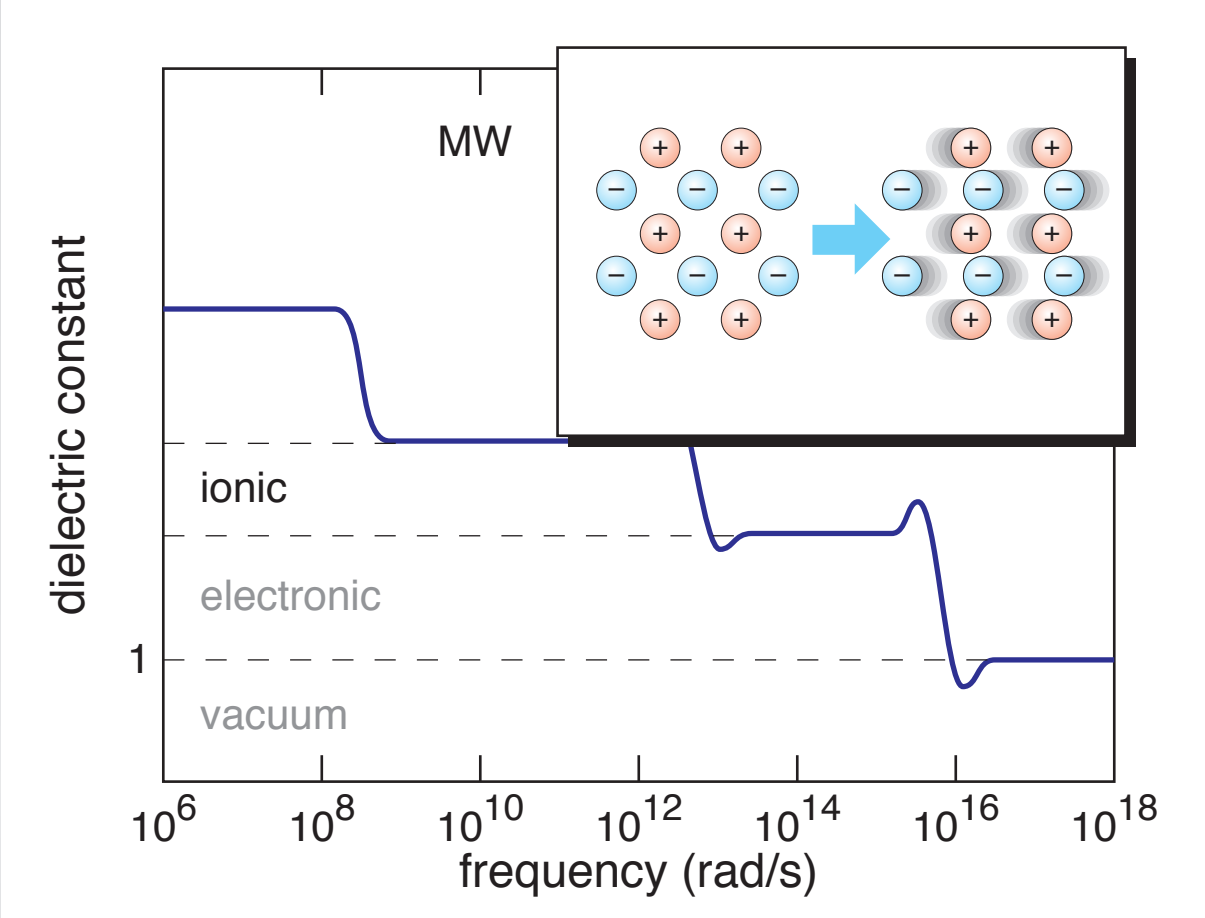

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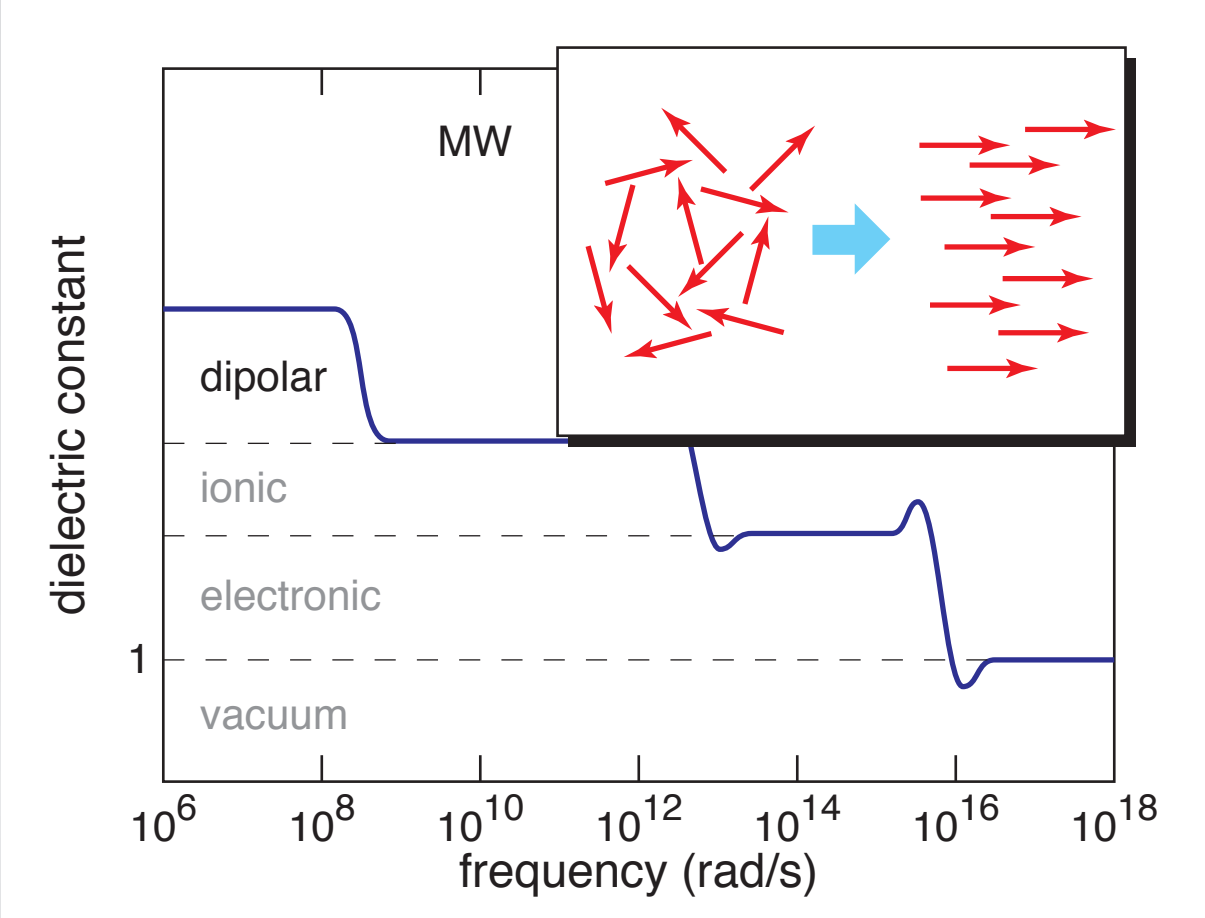

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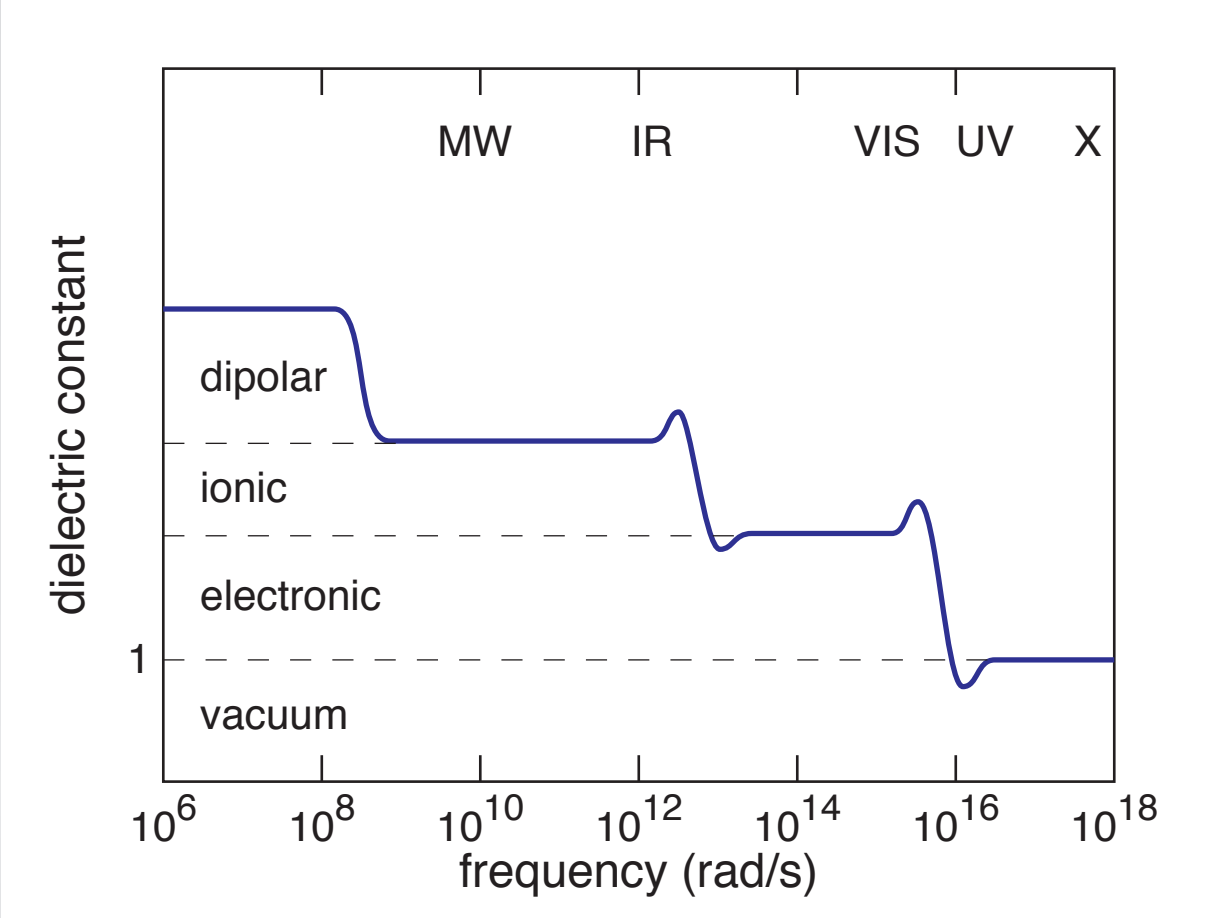

Bound electrons

Electron on a string: $$F_{binding} = -m_e \omega_o^2 x$$

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Steady state: electron oscillates at driving frequency

$$x(t) = x_o e^{-i\omega t}$$

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$$p(t) = -ex(t) = \frac{e^2}{m}\frac{1}{(\omega_o^2 - \omega^2) - i\gamma\omega}E_o e^{-i\omega t}$$

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Polarization

$$P(t) = \left(\frac{Ne^2}{m}\right)\sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - i\gamma_j\omega}E(t) \equiv \epsilon_o \chi_e E(t)$$

Bound electrons

Dielectric function

$$\epsilon(\omega) \equiv 1 + \chi_e = 1 + \frac{Ne^2}{\epsilon_o m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - i\gamma_j\omega}$$

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Q: For a single resonance, is the value of $\epsilon(\omega)$ at high frequency

1. **larger than,**
2. **the same as, or**
3. **smaller than the value at low frequency?**

Bound electrons

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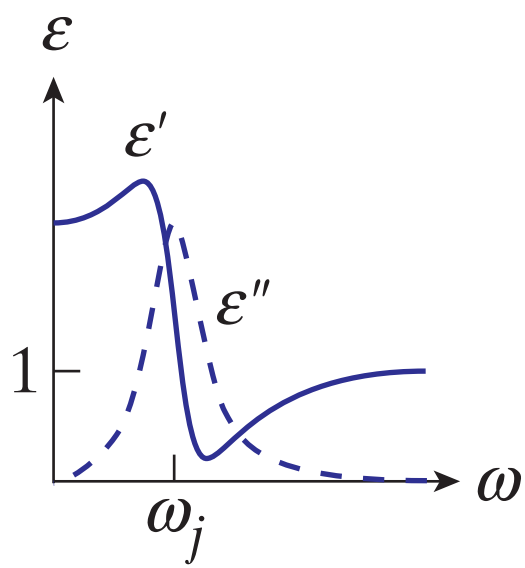
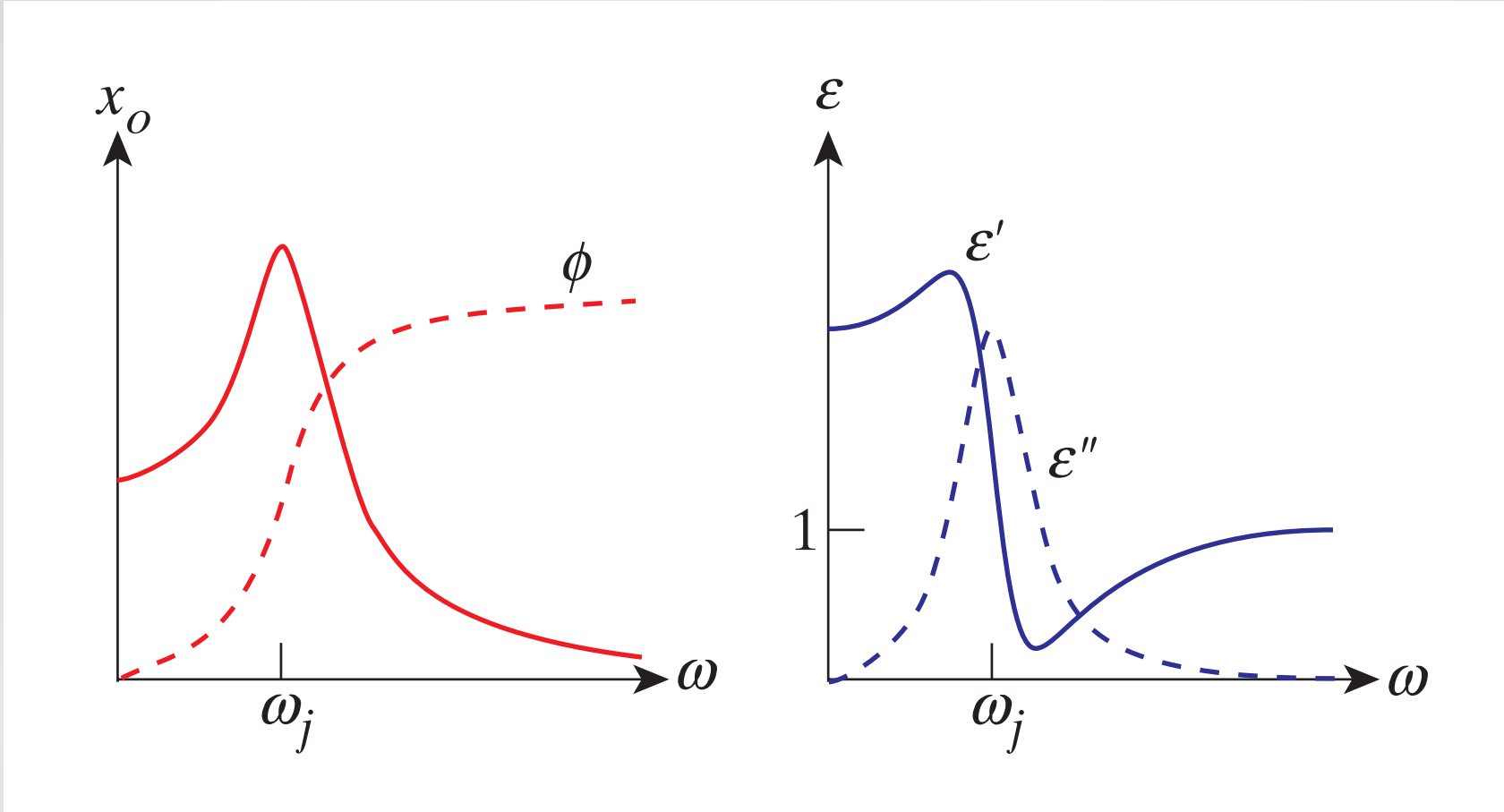


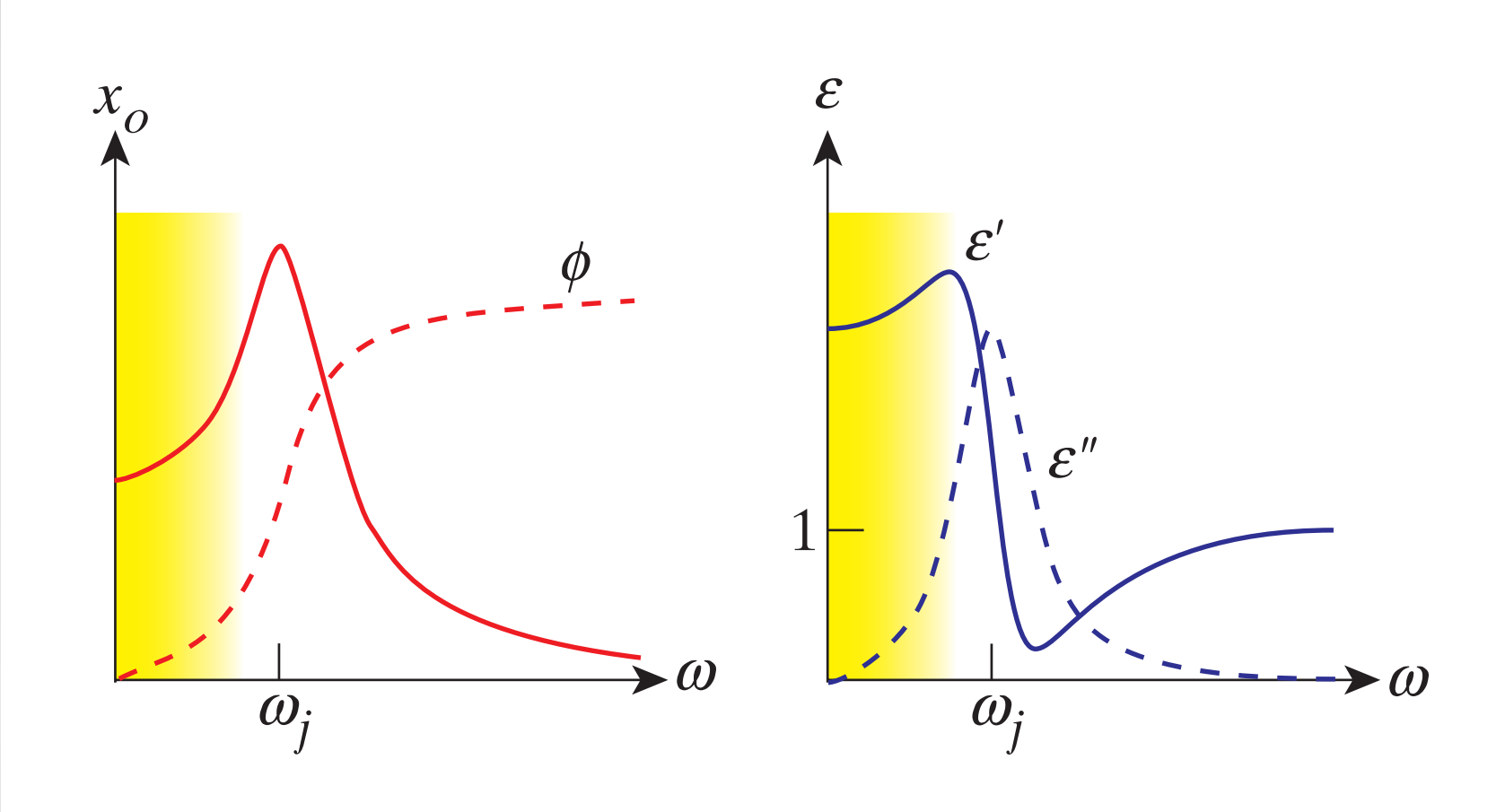
Bound electrons

Amplitude of bound charge oscillation



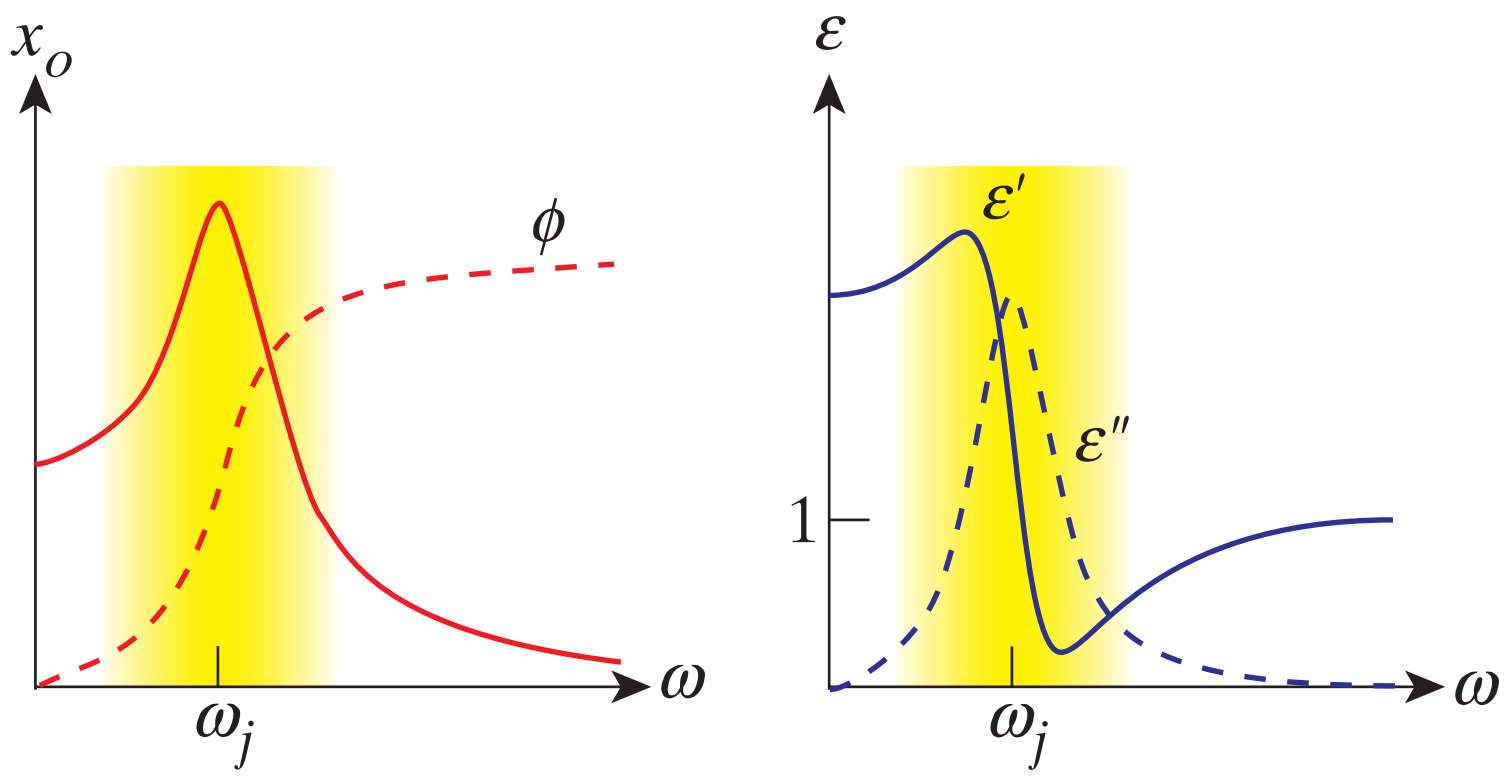
Bound electrons

Below resonance: bound charges keep up with driving field $\Rightarrow$ field attenuated, wave propagates more slowly



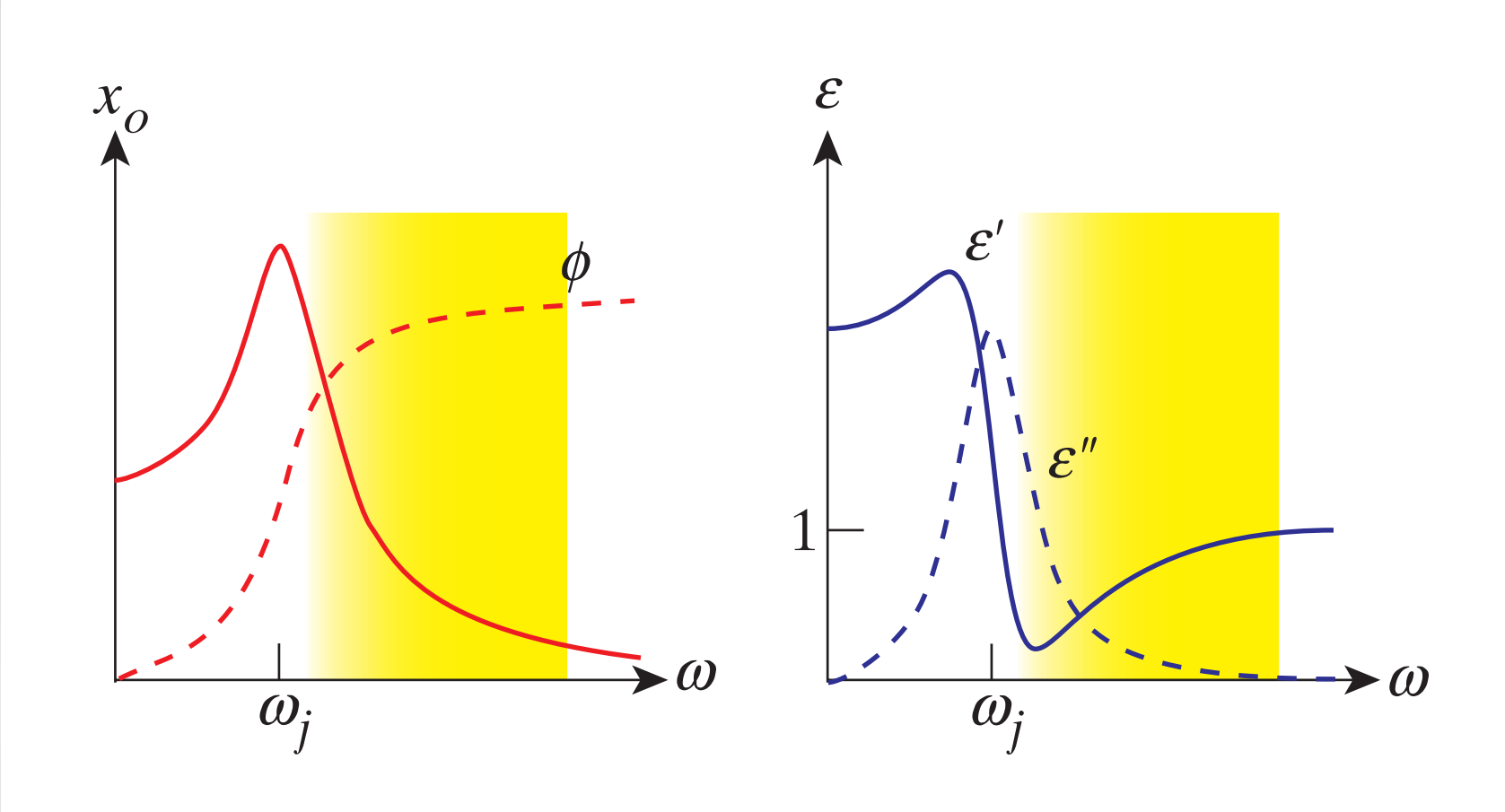
Bound electrons

At resonance: energy transfer from wave to bound charges $\Rightarrow$ wave attenuates (absorption)



Bound electrons

Above resonance: bound charges cannot keep up with driving field $\Rightarrow$ dielectric like a vacuum



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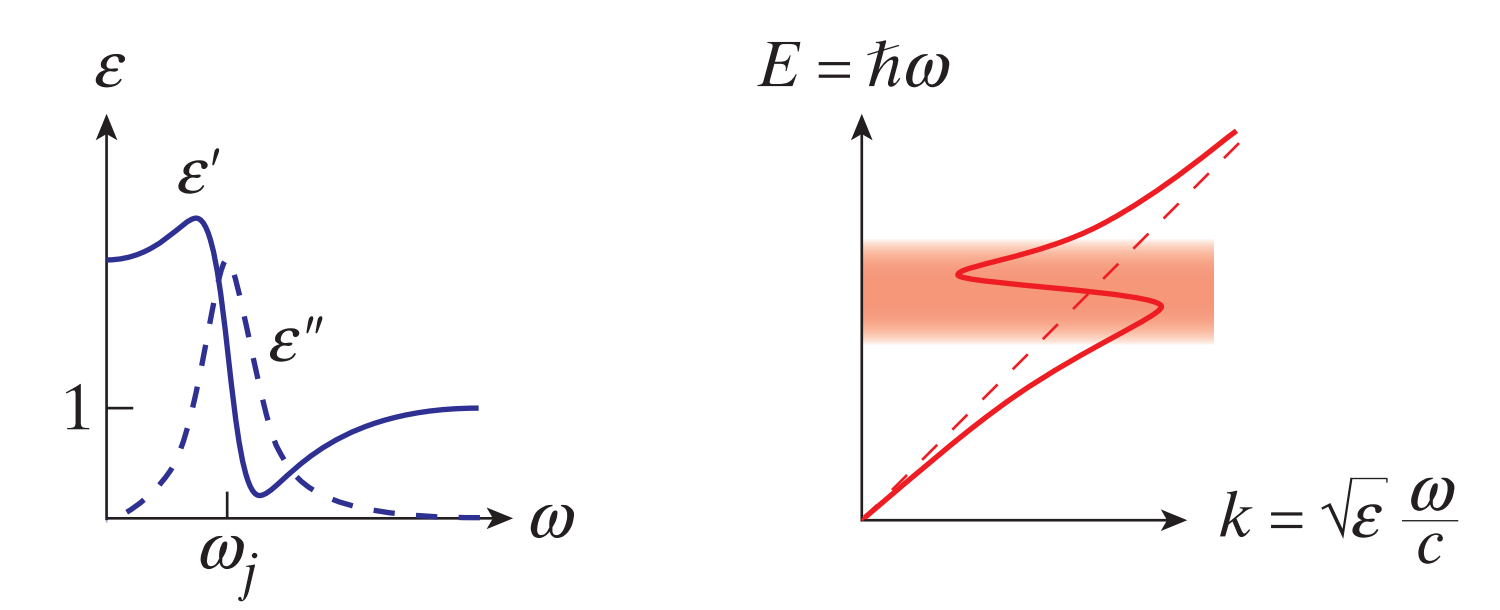

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Low frequency ($\omega \ll \gamma$) $\Rightarrow$ current generated

$$J = -Ne\frac{dx}{dt} = \frac{Ne^2}{m}\frac{1}{\gamma - i\omega}E \approx \frac{Ne^2}{m\gamma}E \equiv \sigma E$$

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$\omega \gg \gamma$: σ complex $\Rightarrow$ J out of phase with E

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$$\epsilon(\omega) \equiv 1 + \chi_e = 1 - \frac{Ne^2}{m\epsilon_o}\frac{1}{\omega^2 + i\gamma\omega} = \epsilon'(\omega) + i\epsilon''(\omega)$$

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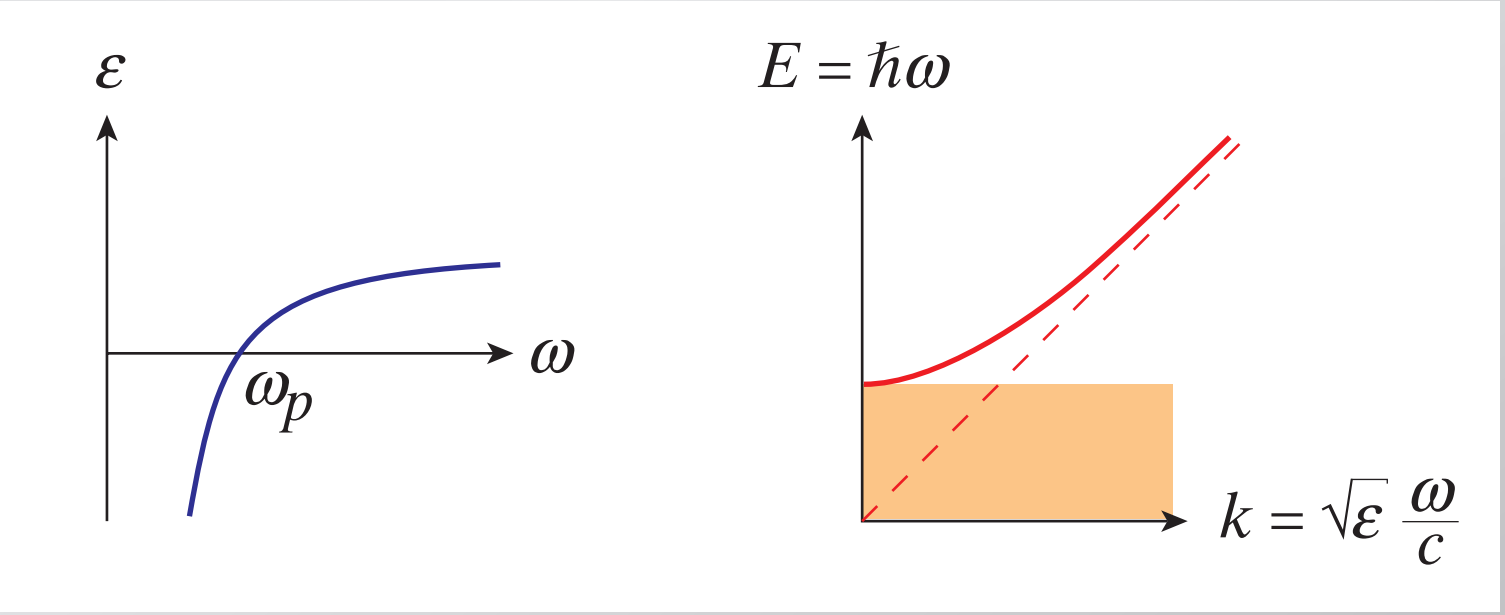

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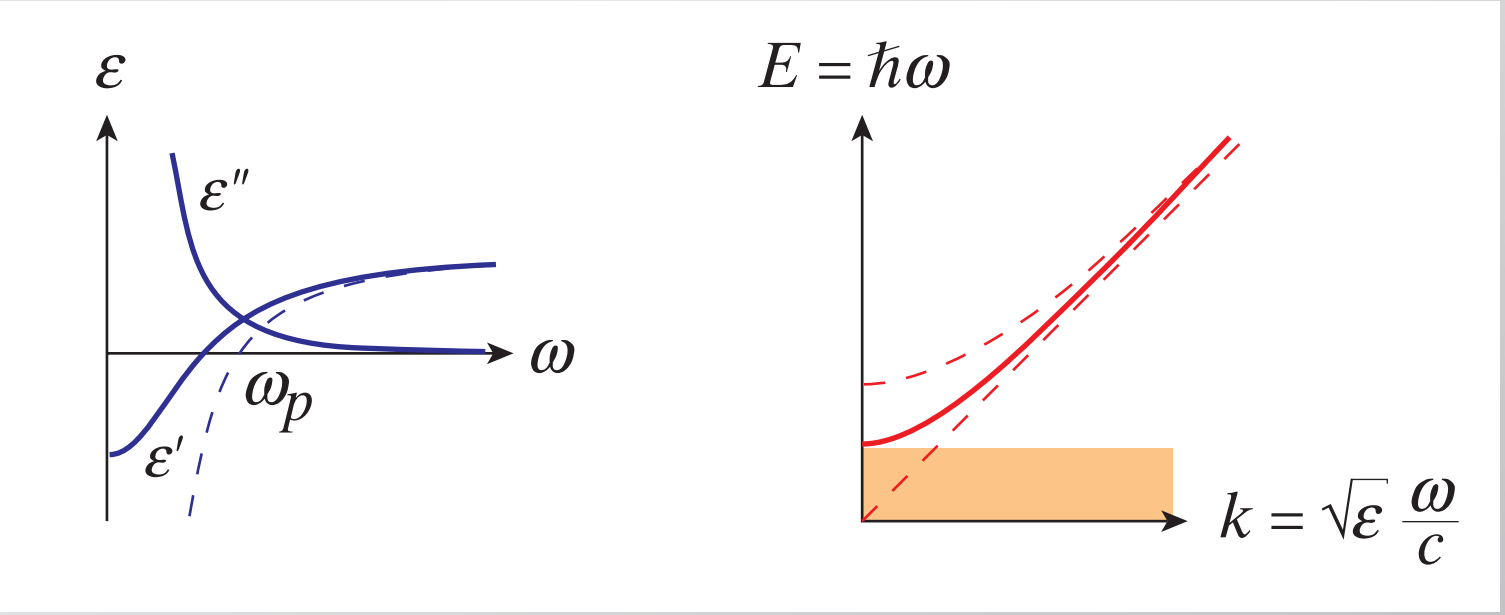
Add damping:

$$\gamma \lesssim \omega_p$$



Free electrons

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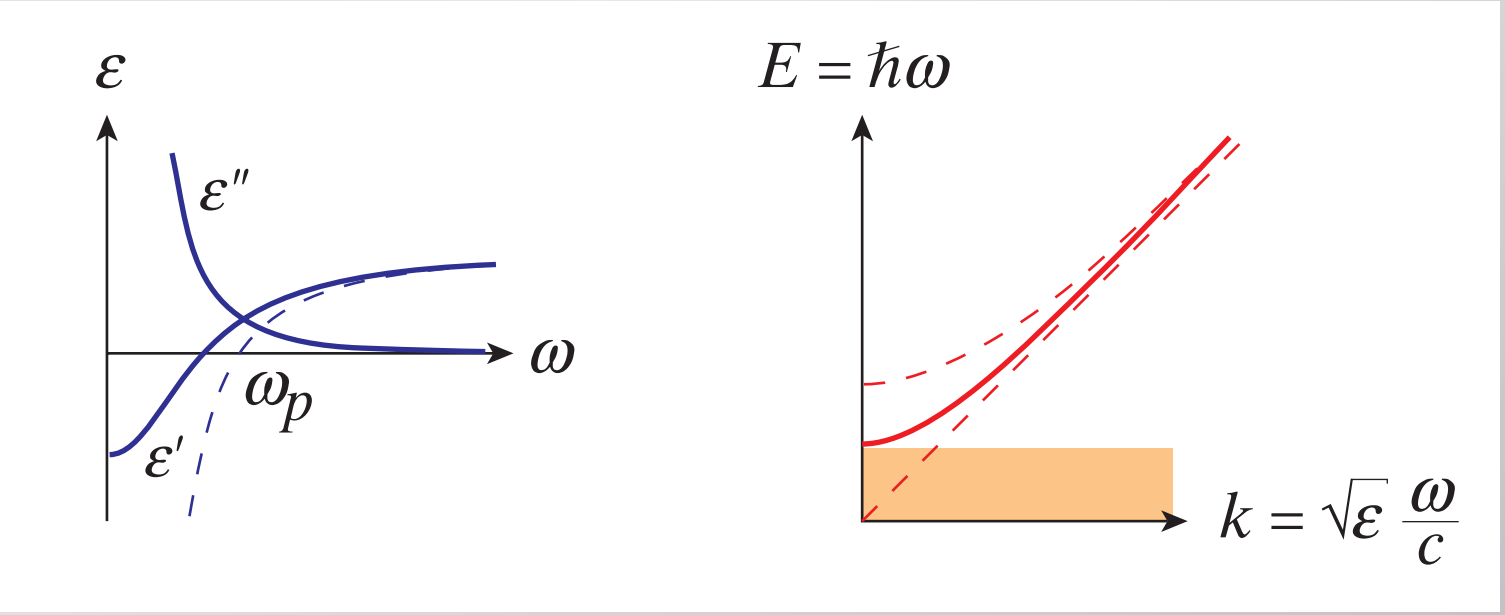

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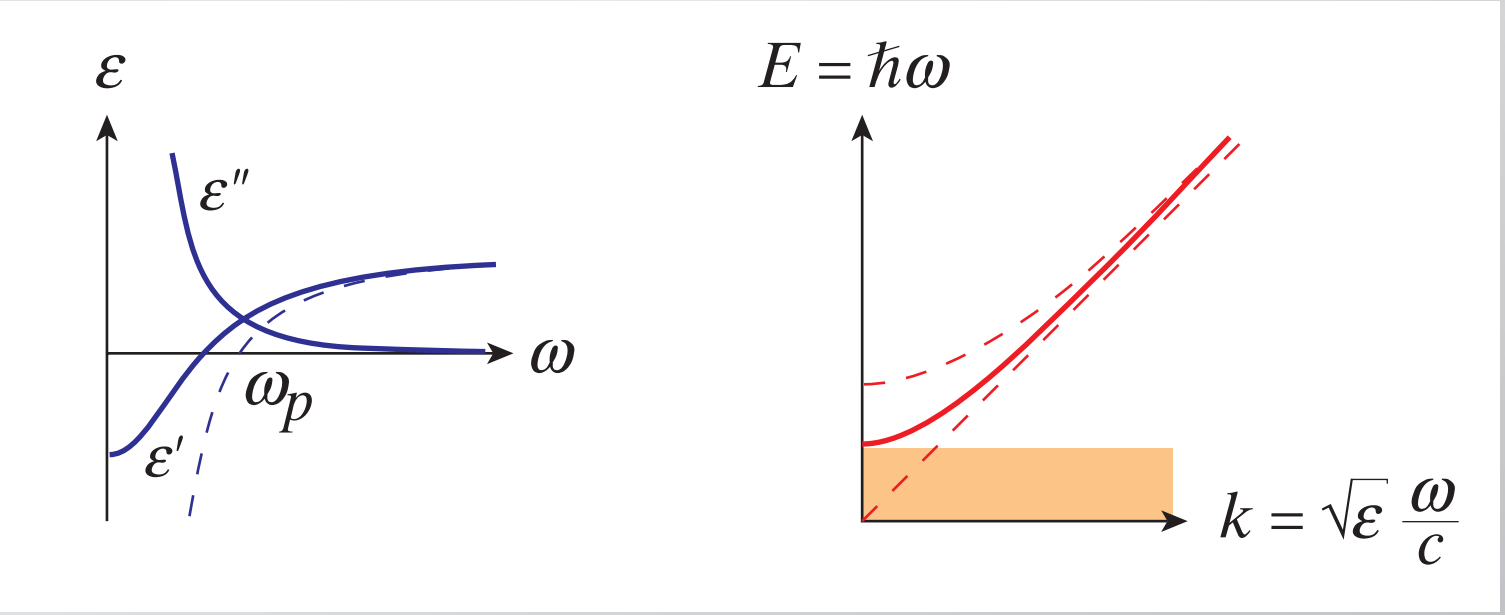

Free electrons

Q: When the wavelength becomes infinite, the index…

1. becomes very large
2. is zero
3. becomes smaller than 1 (but remains positive)
4. becomes negative

Free electrons

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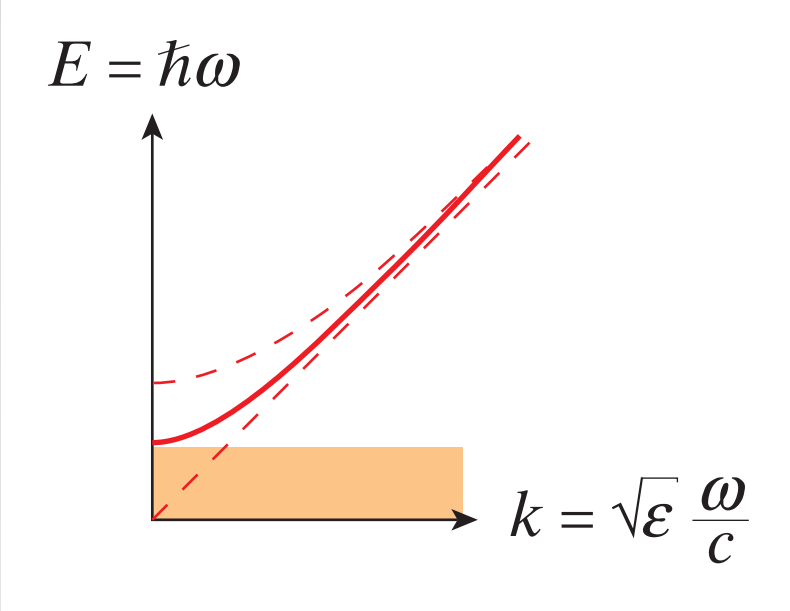
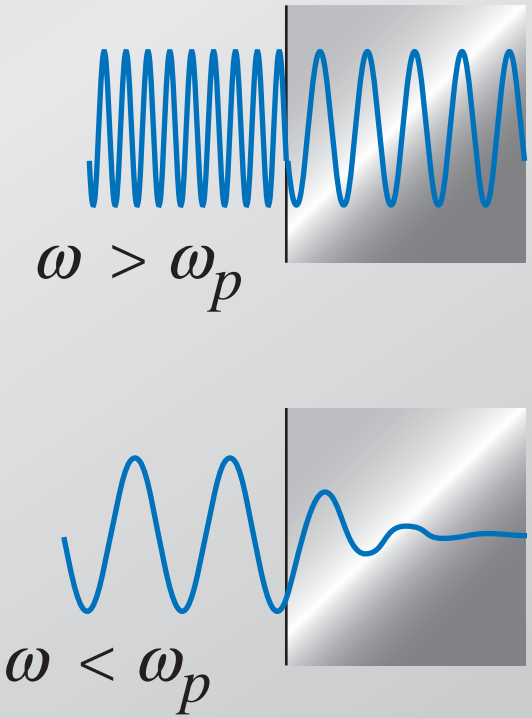
Free electrons

Plasma acts like a high-pass filter

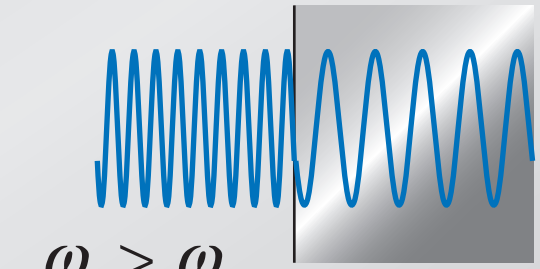




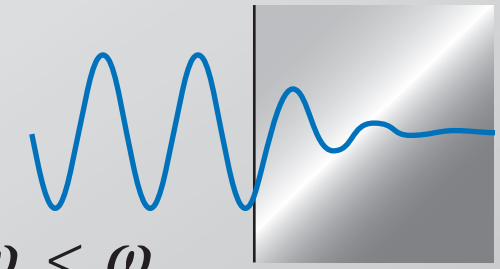
Free electrons

Plasma acts like a high-pass filter

log N (cm^{-3})	ω_p (rad s^{-1})	λ_p
22	6×10^{15}	330 nm
18	6×10^{13}	33 μm
14	6×10^{11}	3.3 mm
10	6×10^{9}	0.33 m



$\omega > \omega_p$



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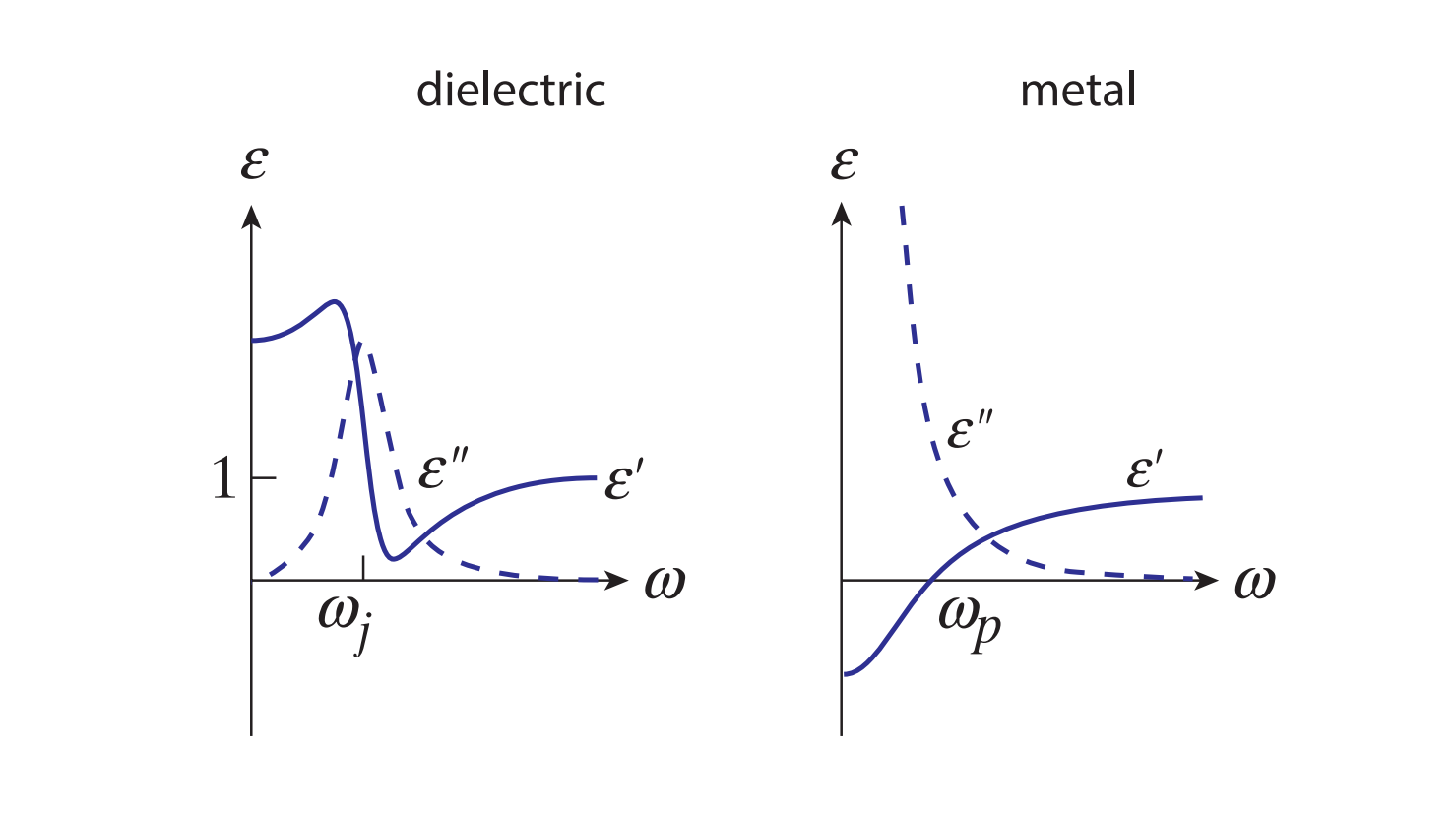
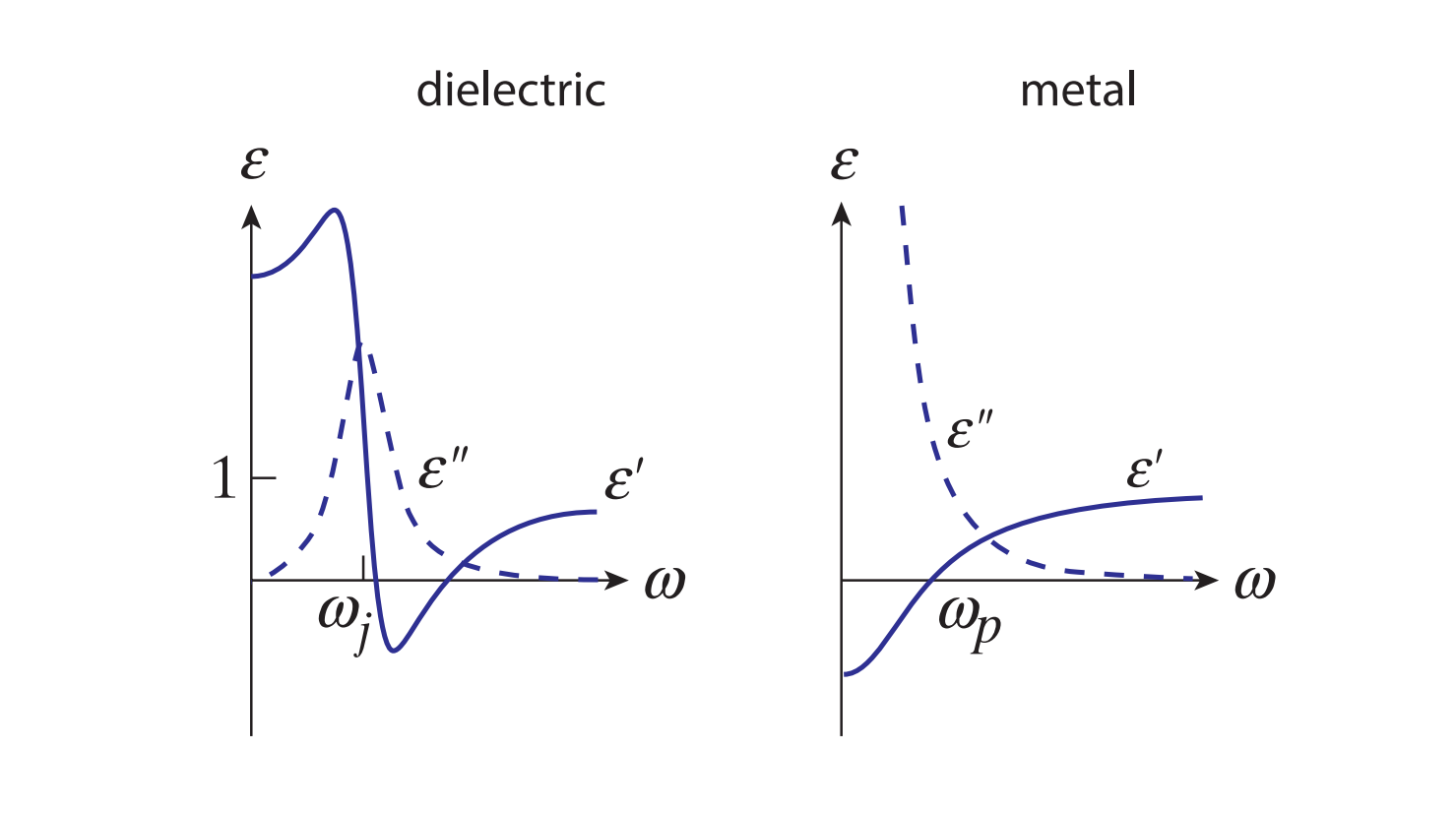
Dielectrics as conductors



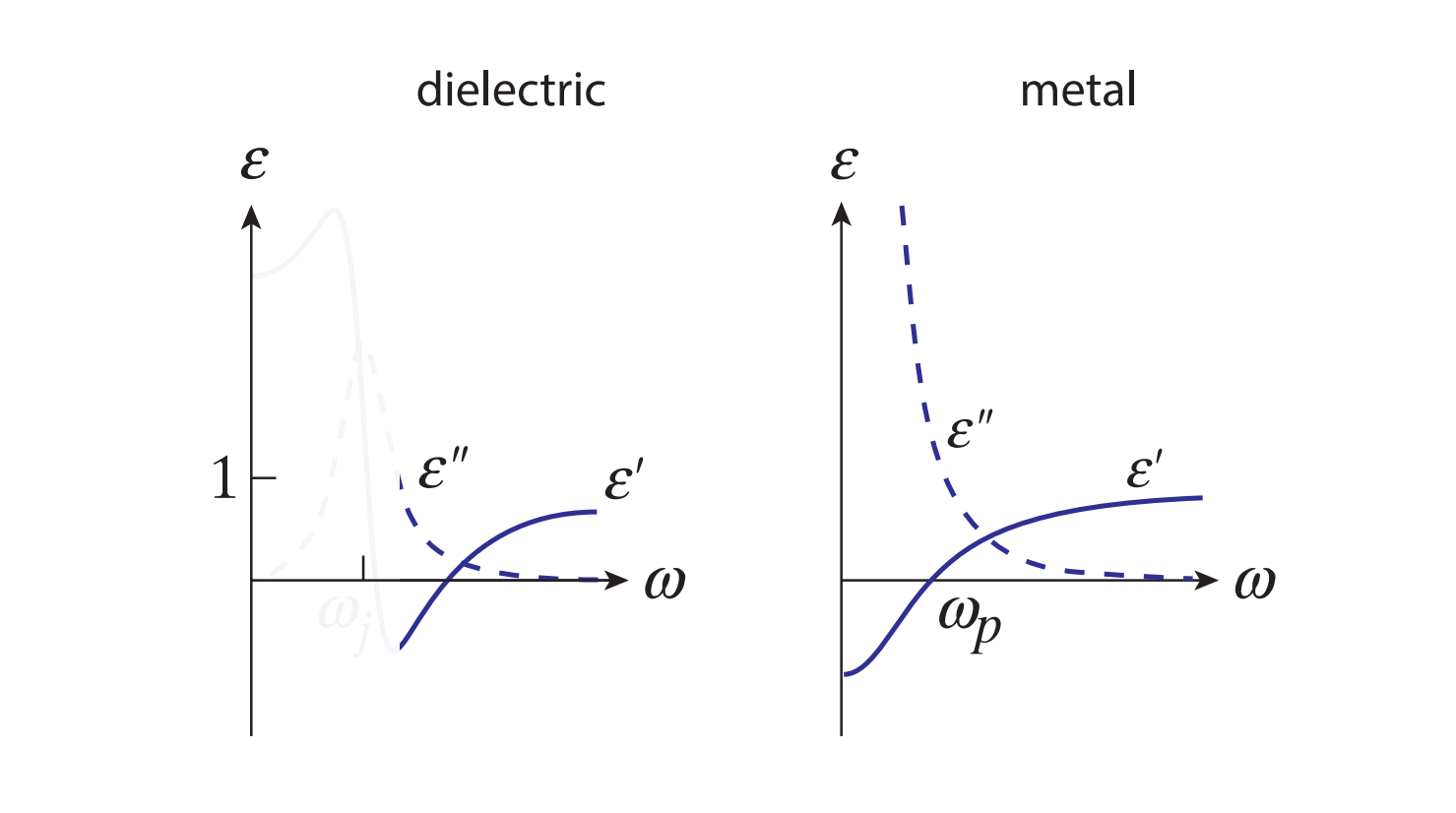
Dielectrics as conductors

for a strong (dielectric) resonance ε can become negative



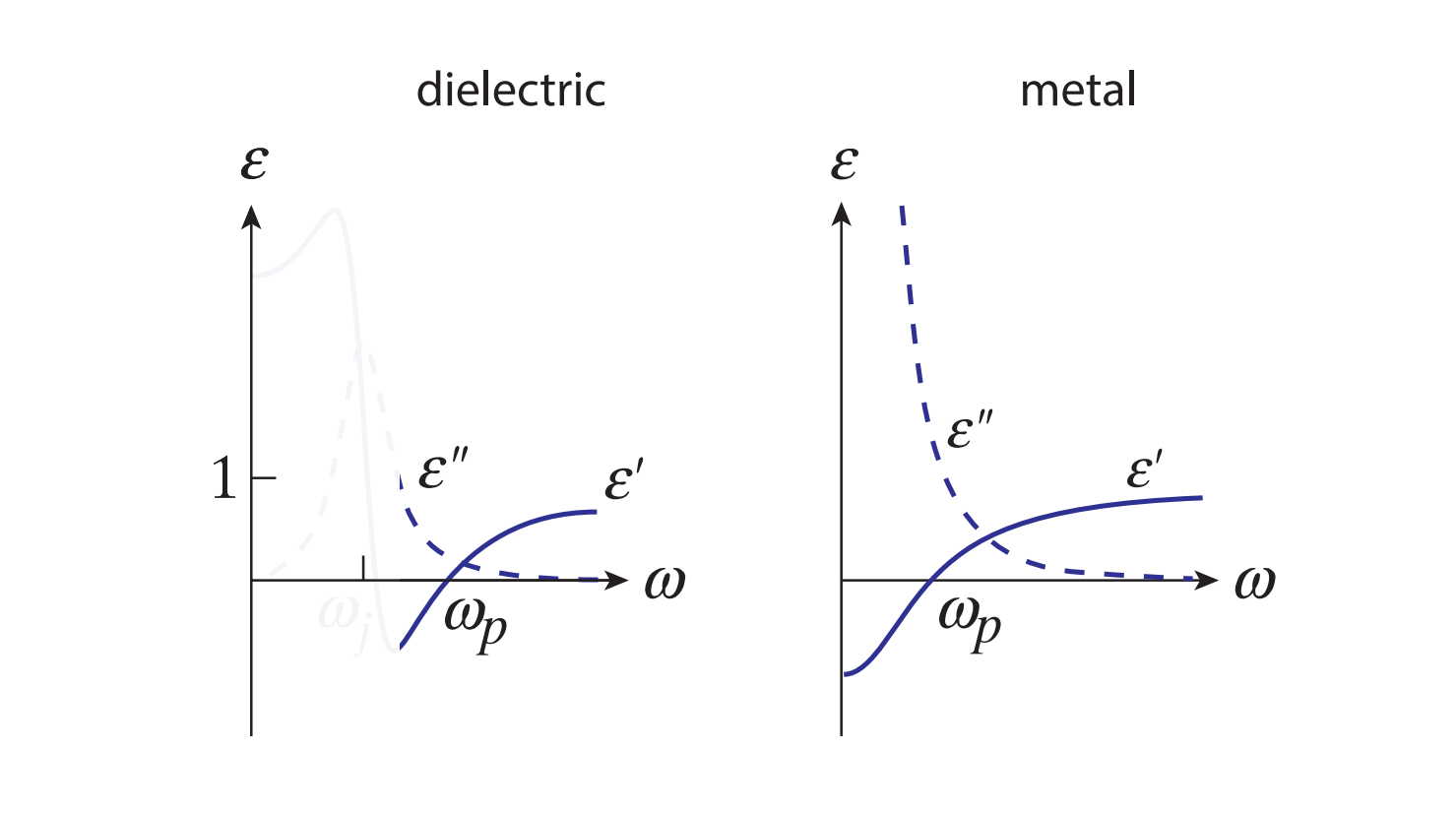
Dielectrics as conductors

valence electrons in dielectric then behave like a plasma



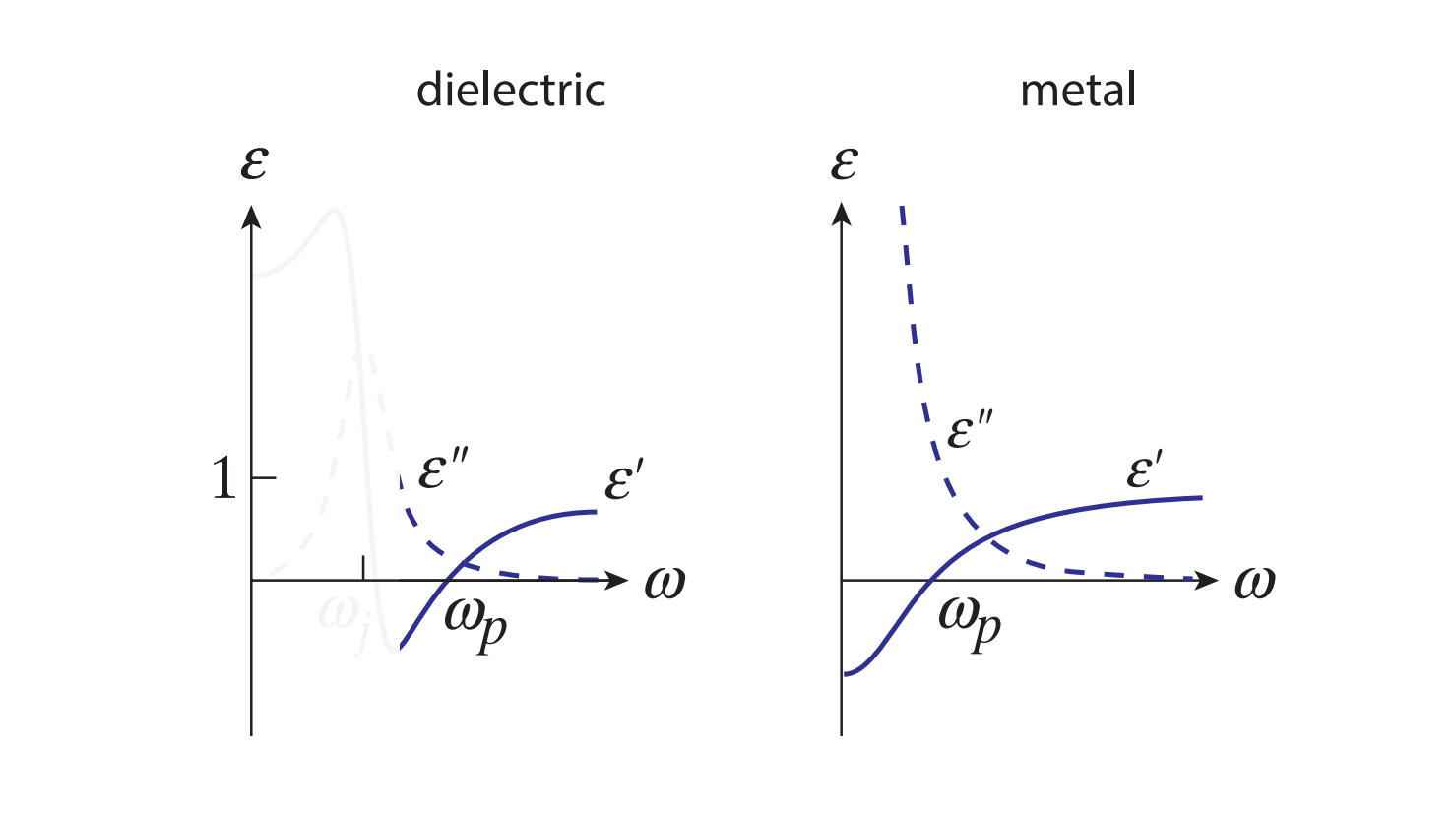
Dielectrics as conductors

with plasma freqency above the resonance



Dielectrics as conductors

(and far below the UV region)



Magnetic response

Index also determined by magnetic response

$$n = \sqrt{\epsilon\mu}$$

Magnetic response

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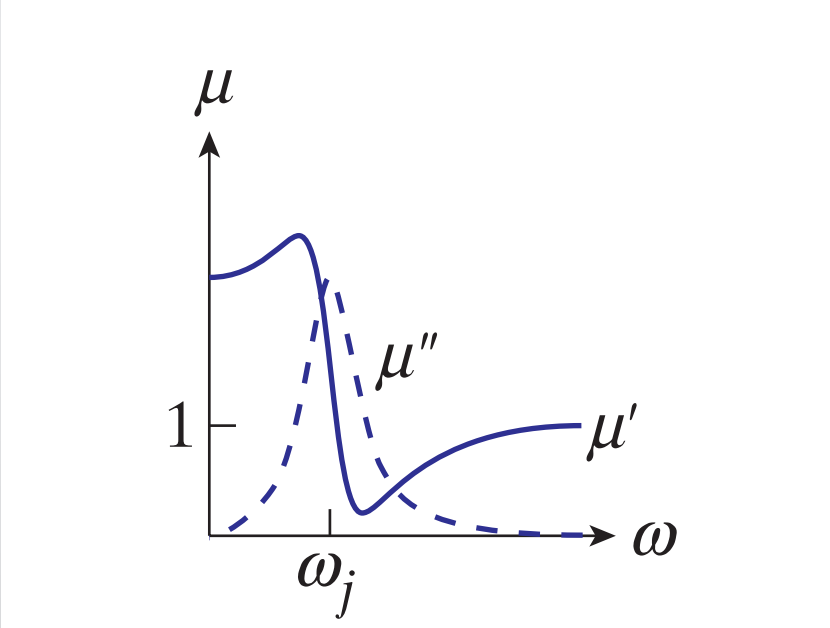
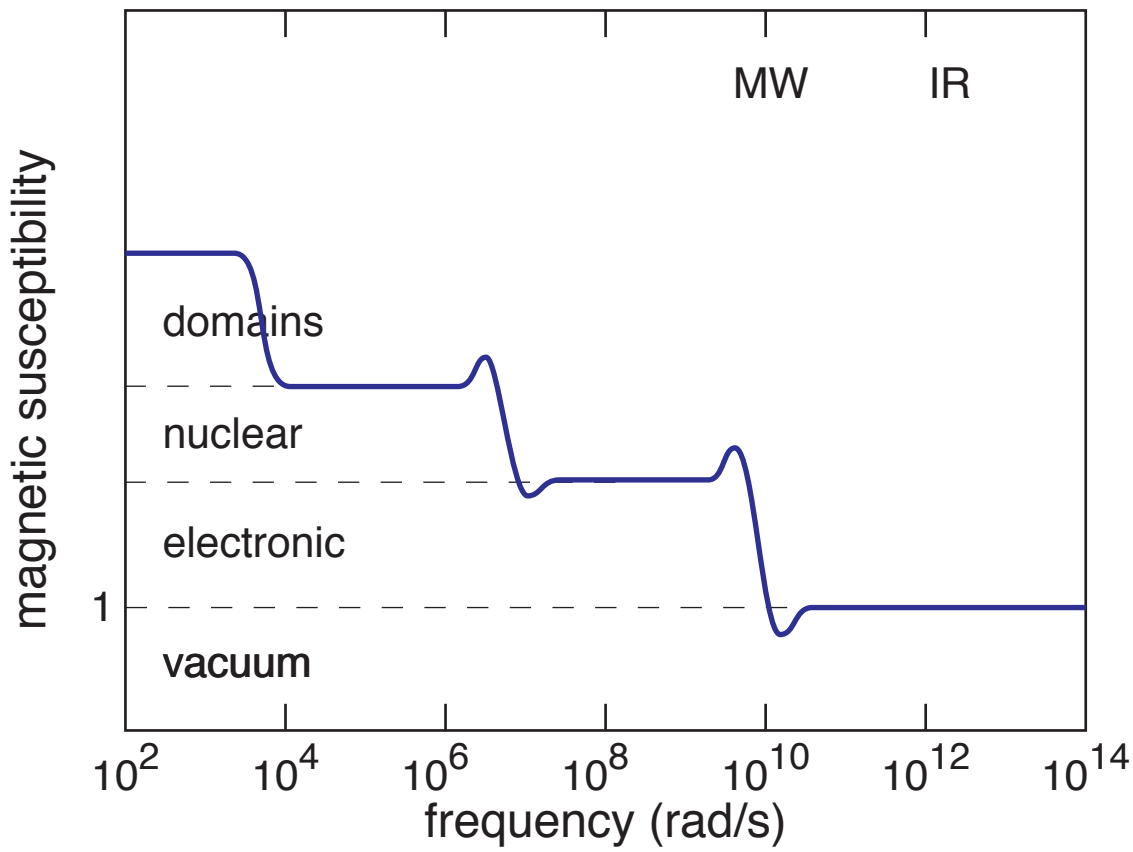
and magnetic response shows similar resonances



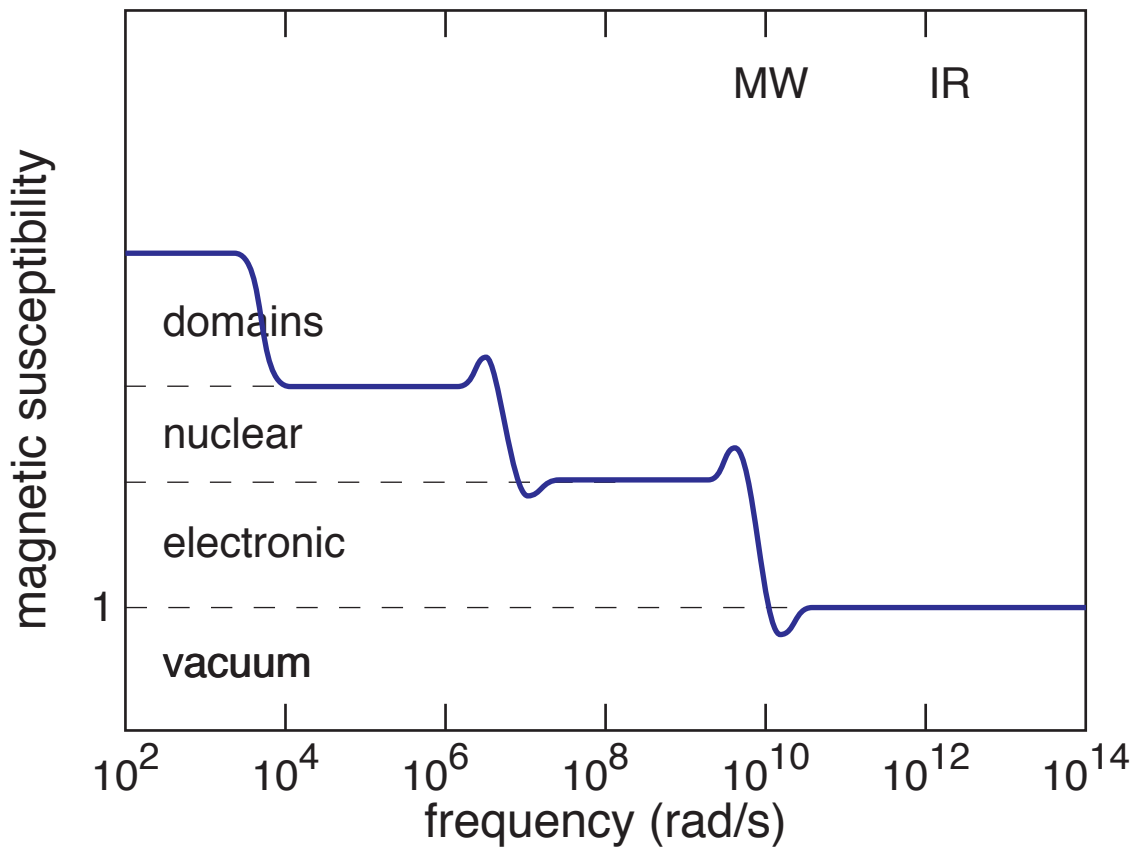
Magnetic response

but magnetic resonances occur below optical frequencies



Magnetic response

so, in optical regime, $\mu \approx 1$



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Index of refraction

$$n = \sqrt{\epsilon\mu}$$

Both ϵ and μ are complex and their real parts can be negative.

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Both ϵ and μ are complex and their real parts can be negative.

What happens when $\mathrm{Re}\epsilon$ and/or $\mathrm{Re}\mu$ is negative?

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Write complex quantities as

$$\varepsilon = |\varepsilon| e^{i\theta} \qquad \mu = |\mu| e^{i\phi}$$

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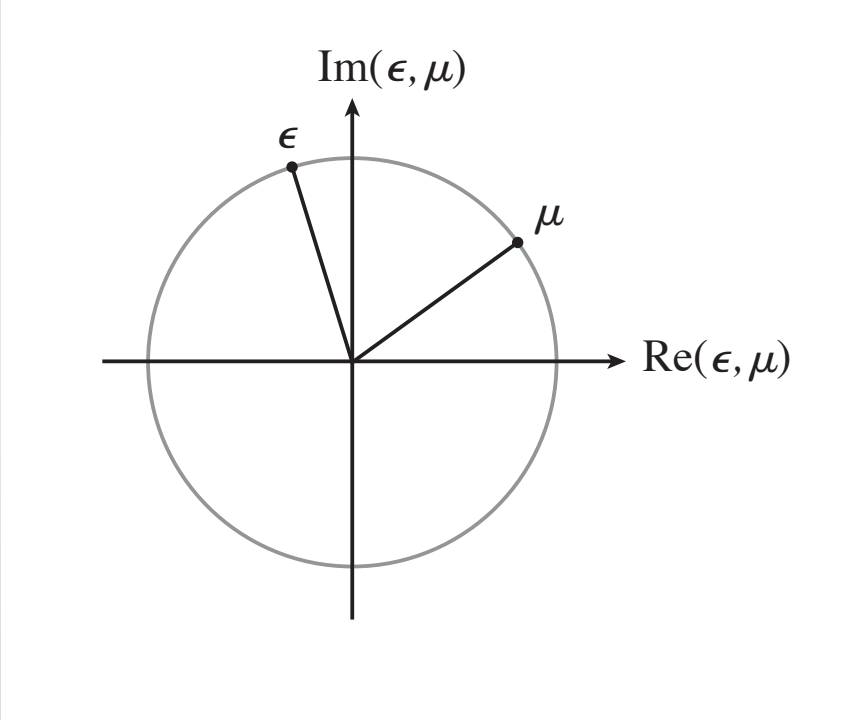

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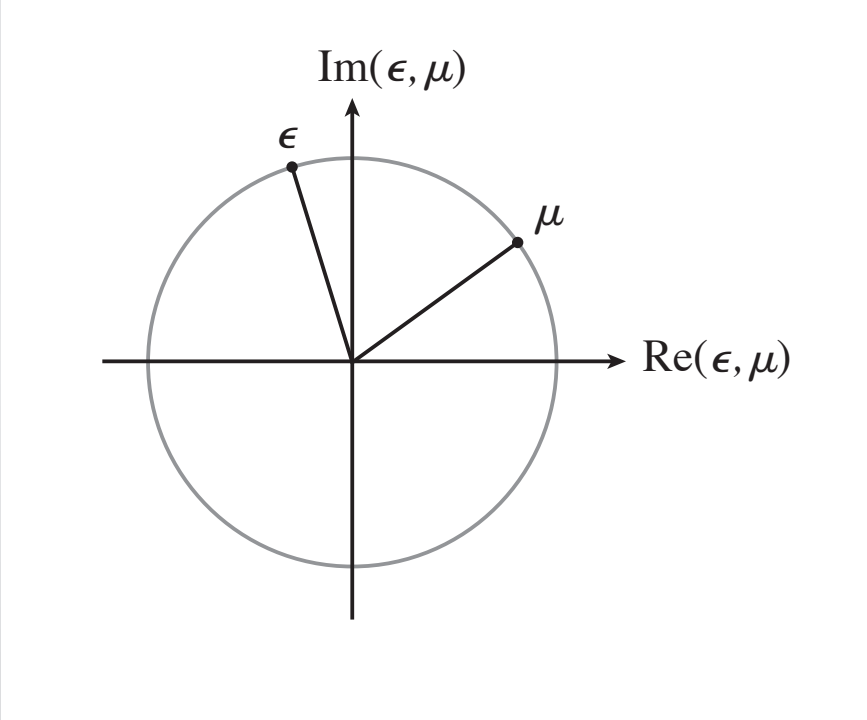

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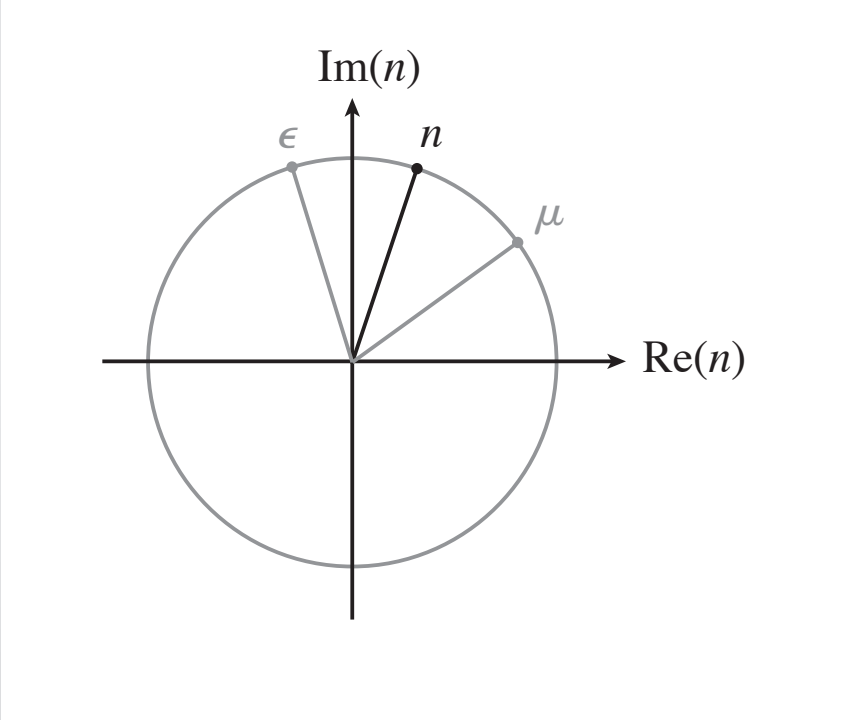

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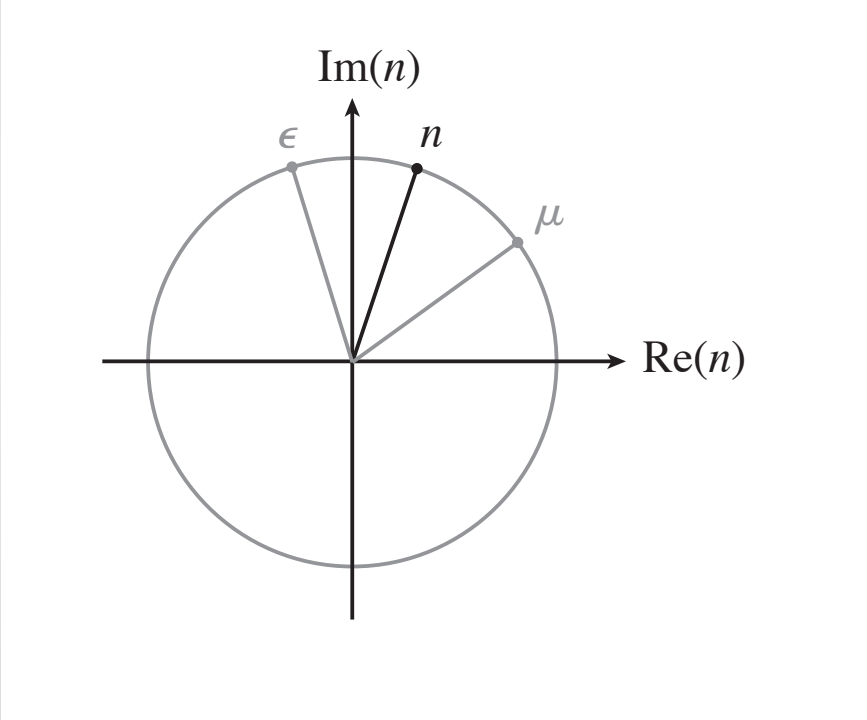

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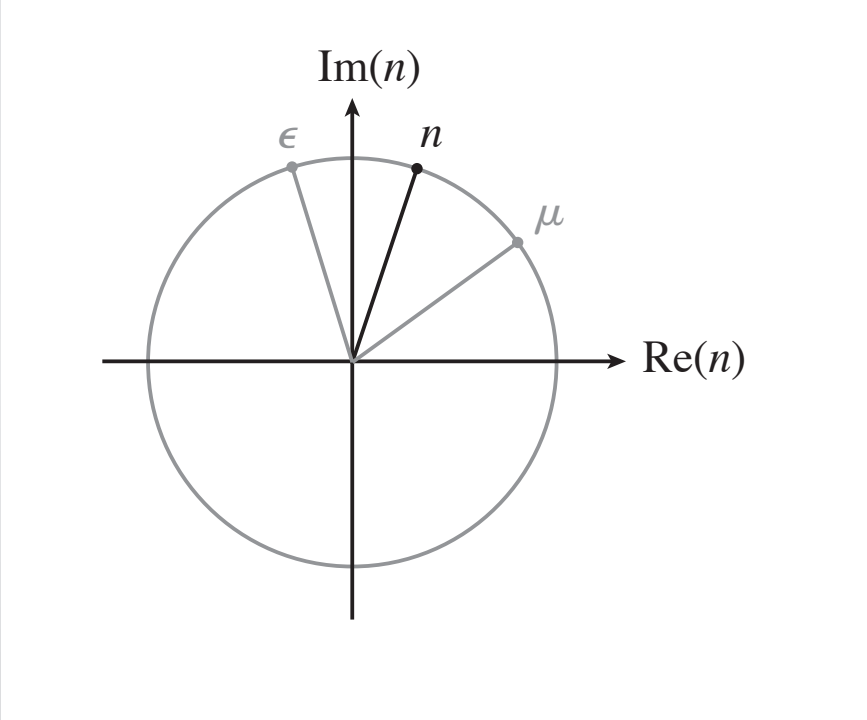

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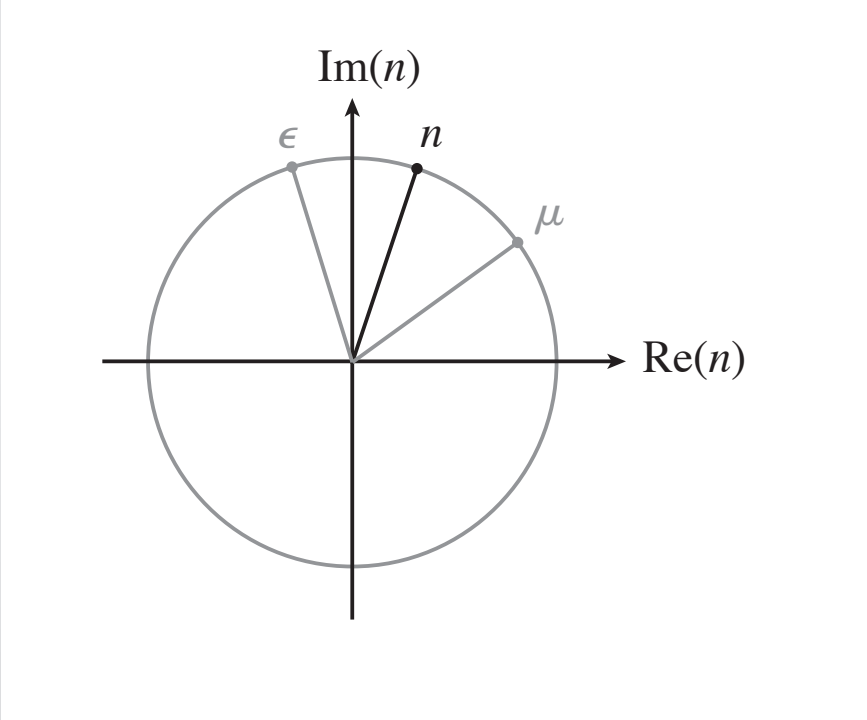

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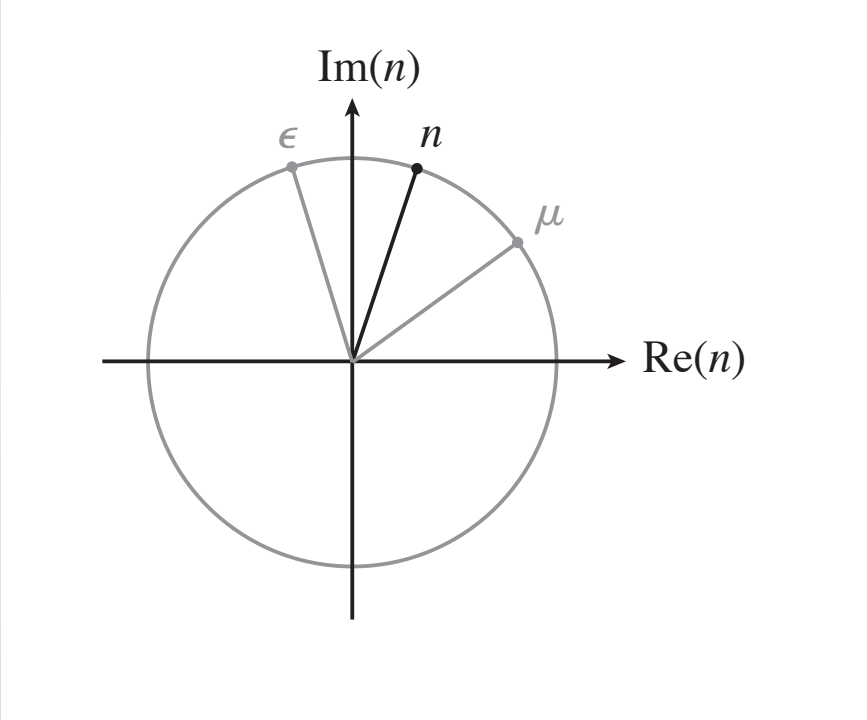

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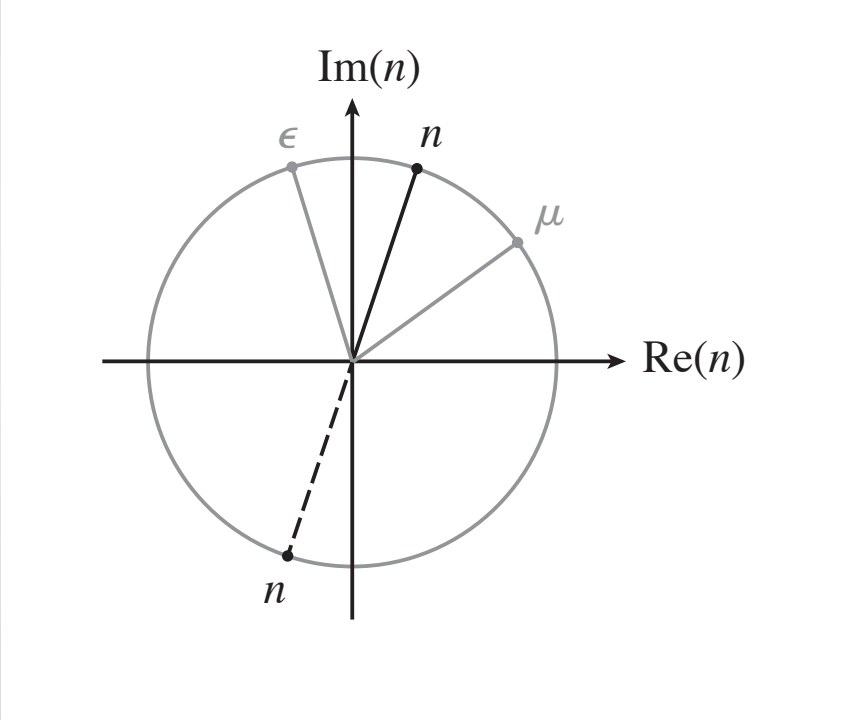

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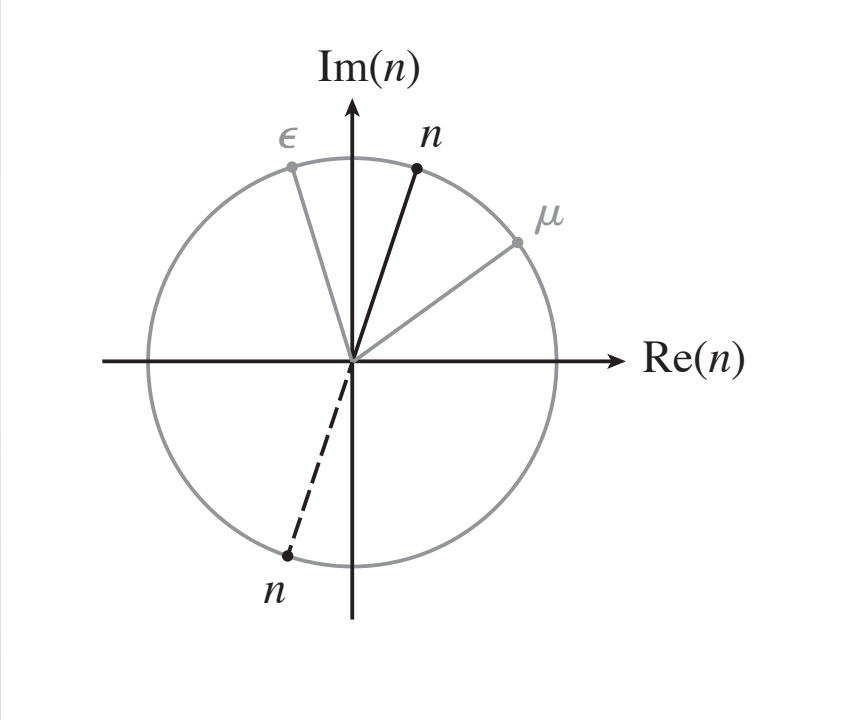

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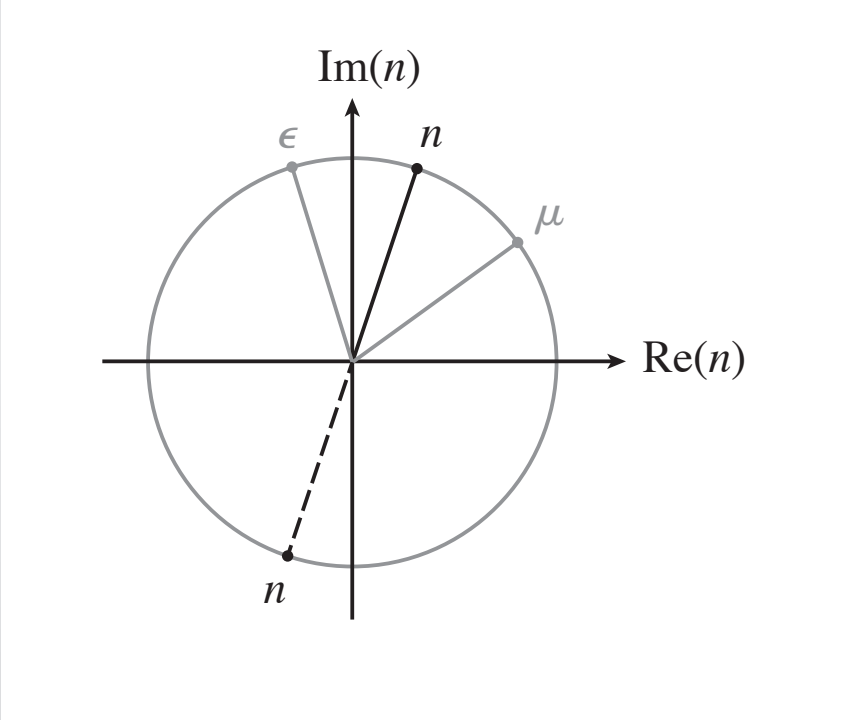

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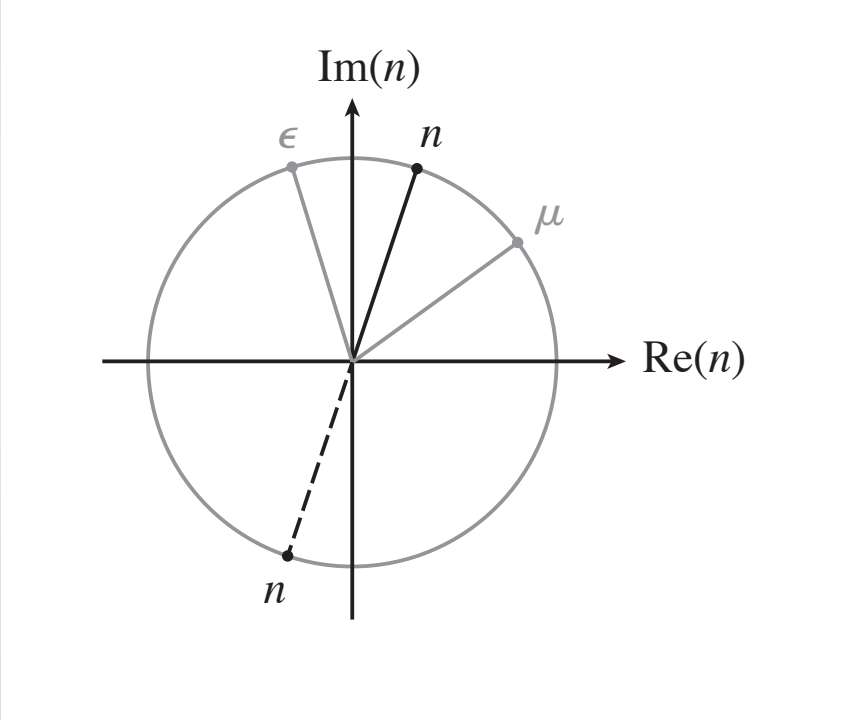

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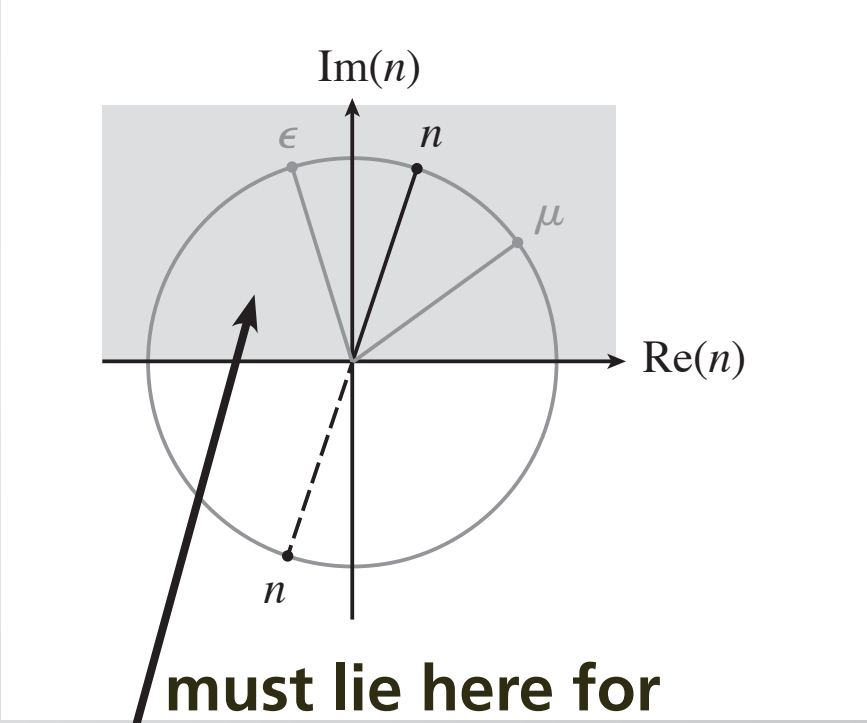

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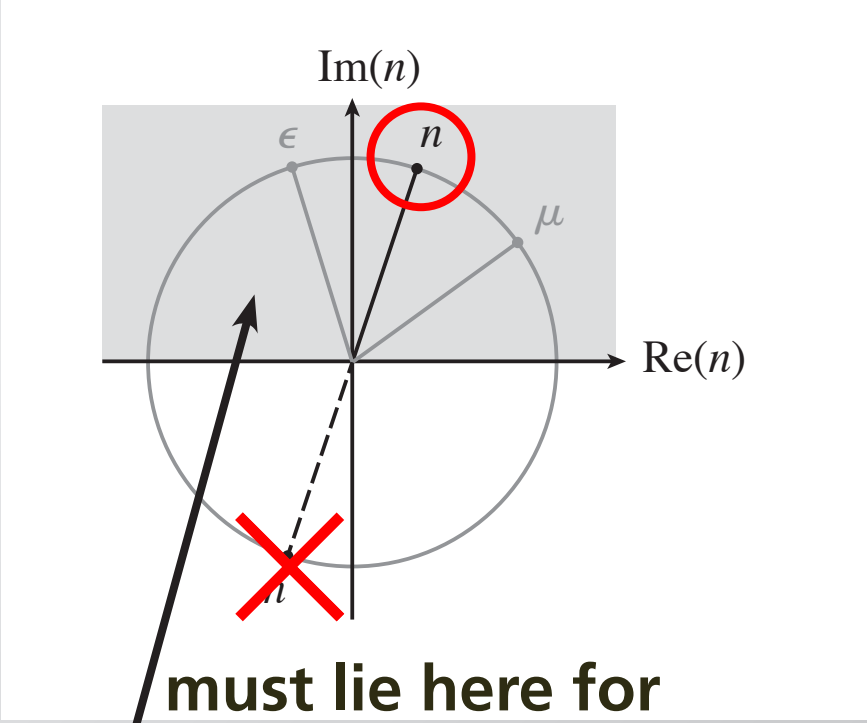

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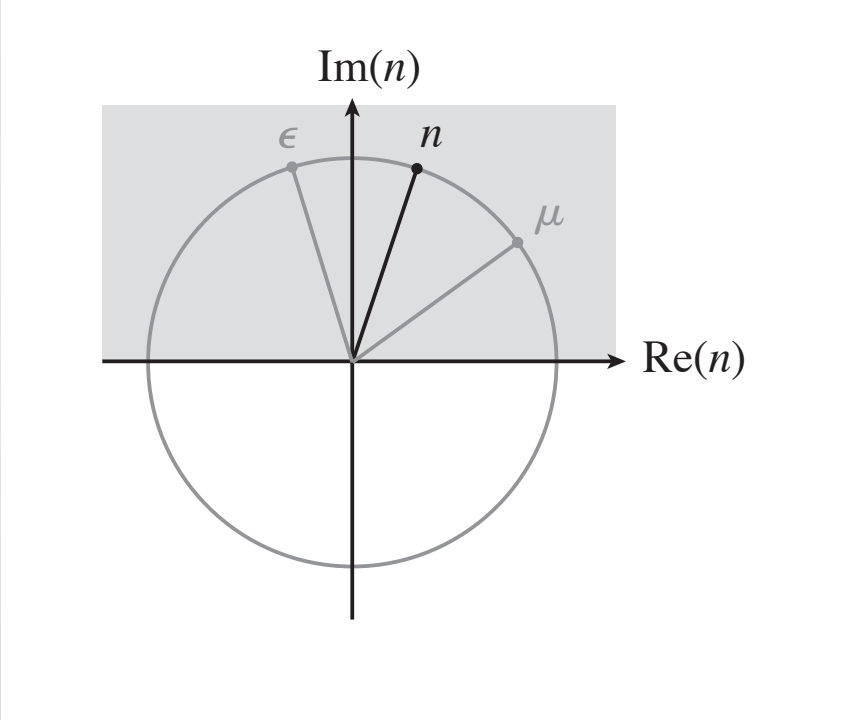

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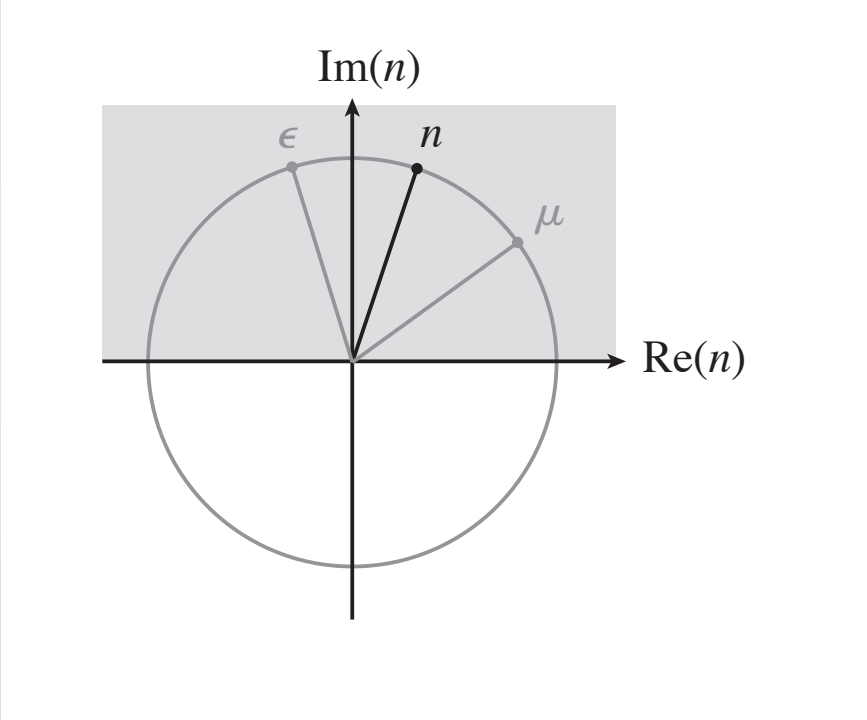
To find n (passive materials):

1. Draw line that bisects ϵ and μ

2. Choose upper branch



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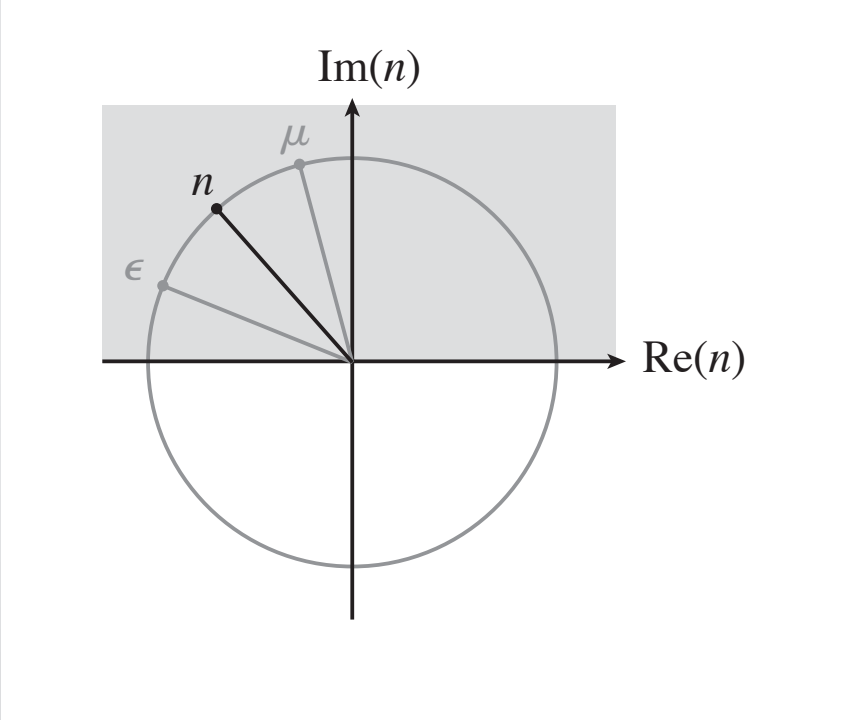
For certain values of ϵ and μ

we can get a *negative* $\mathrm{Re}(n)$!



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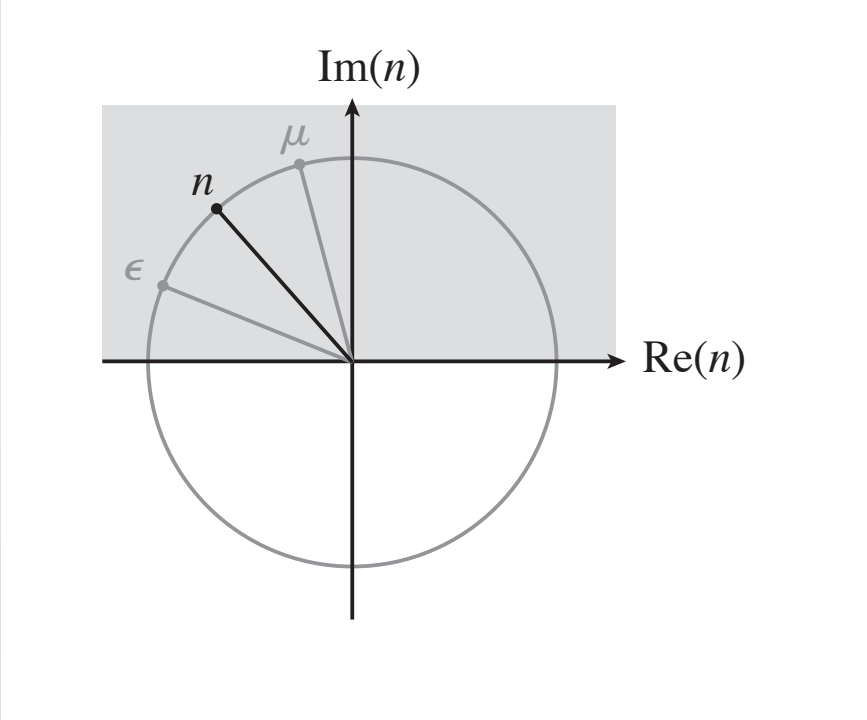
Q: Must both $\mathrm{Re}\epsilon < 0$ and $\mathrm{Re}\mu < 0$ to get a negative index?

1. yes

2. no



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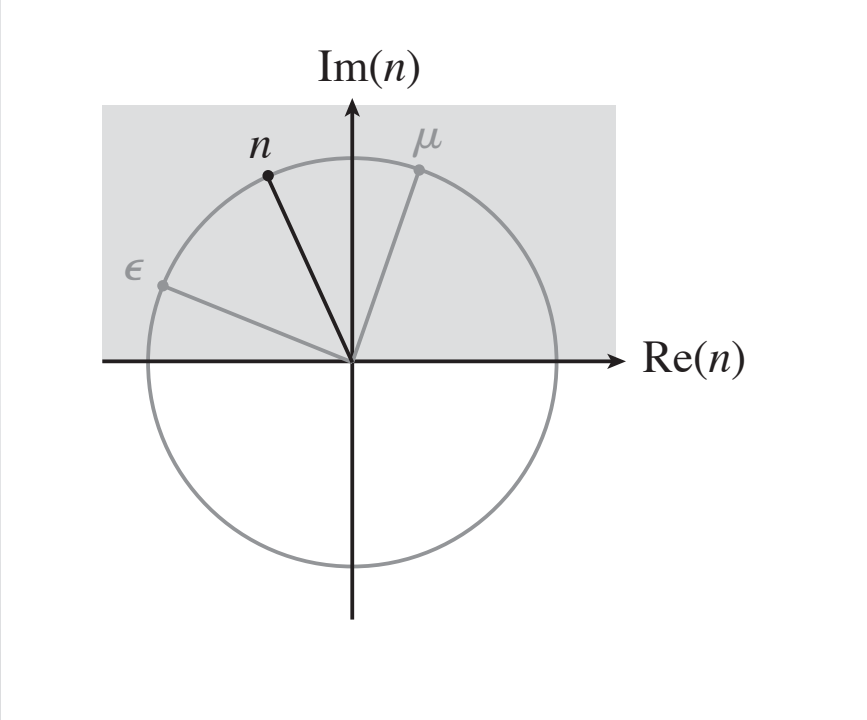

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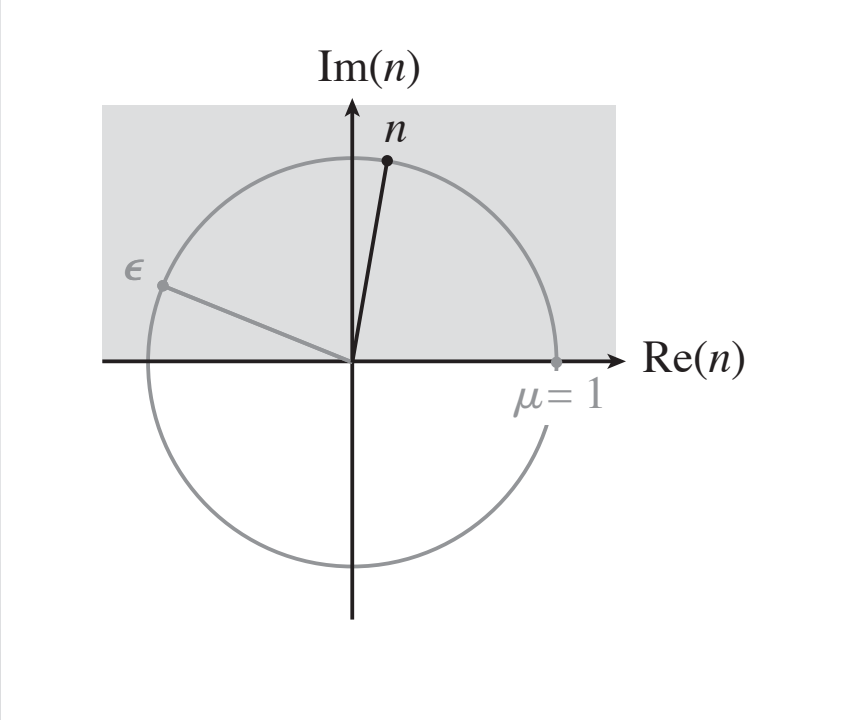
Note: need magnetic response

to achive $n \leq 0$!



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$$E = E_o e^{i(kx - \omega t)} = E_o e^{-k''x} e^{i(k'x - \omega t)}$$

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When $\mathrm{Re}(n) < 0,\ k' < 0$, **and so phase velocity reversed!**

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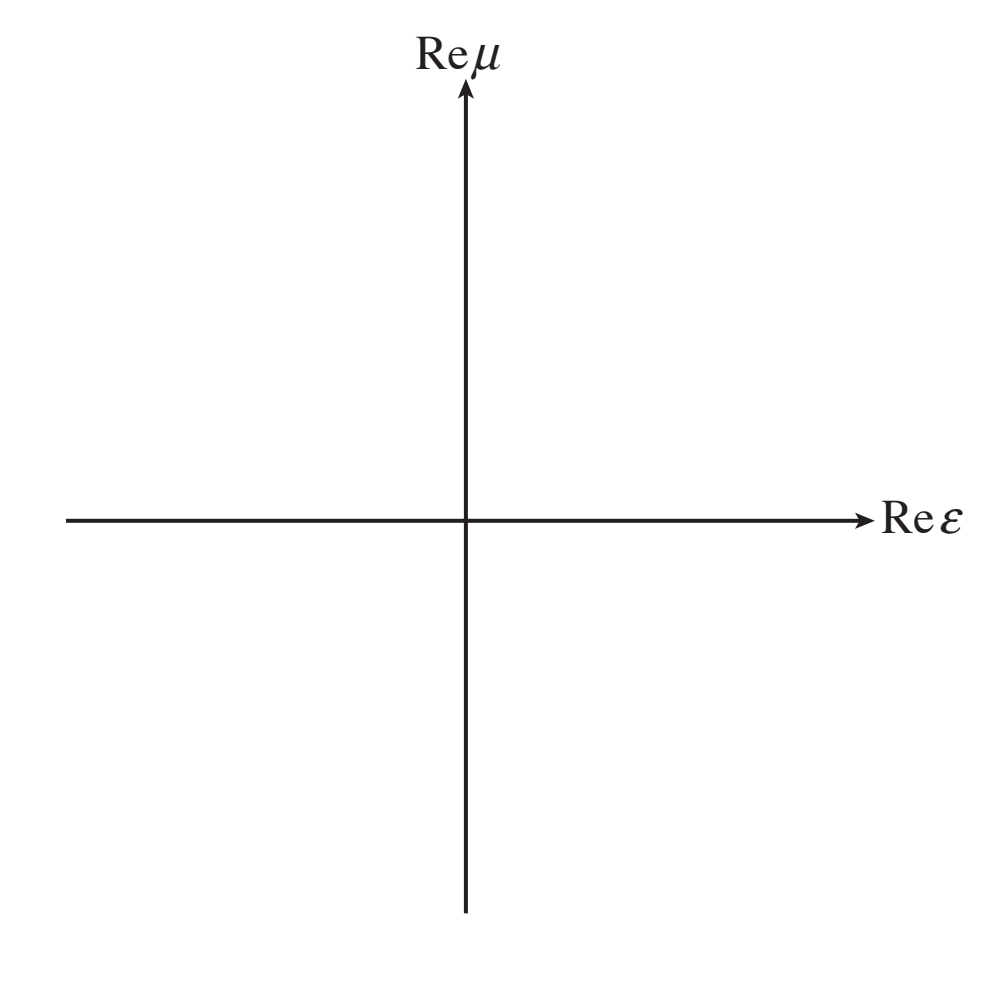
classification of (non-lossy) materials



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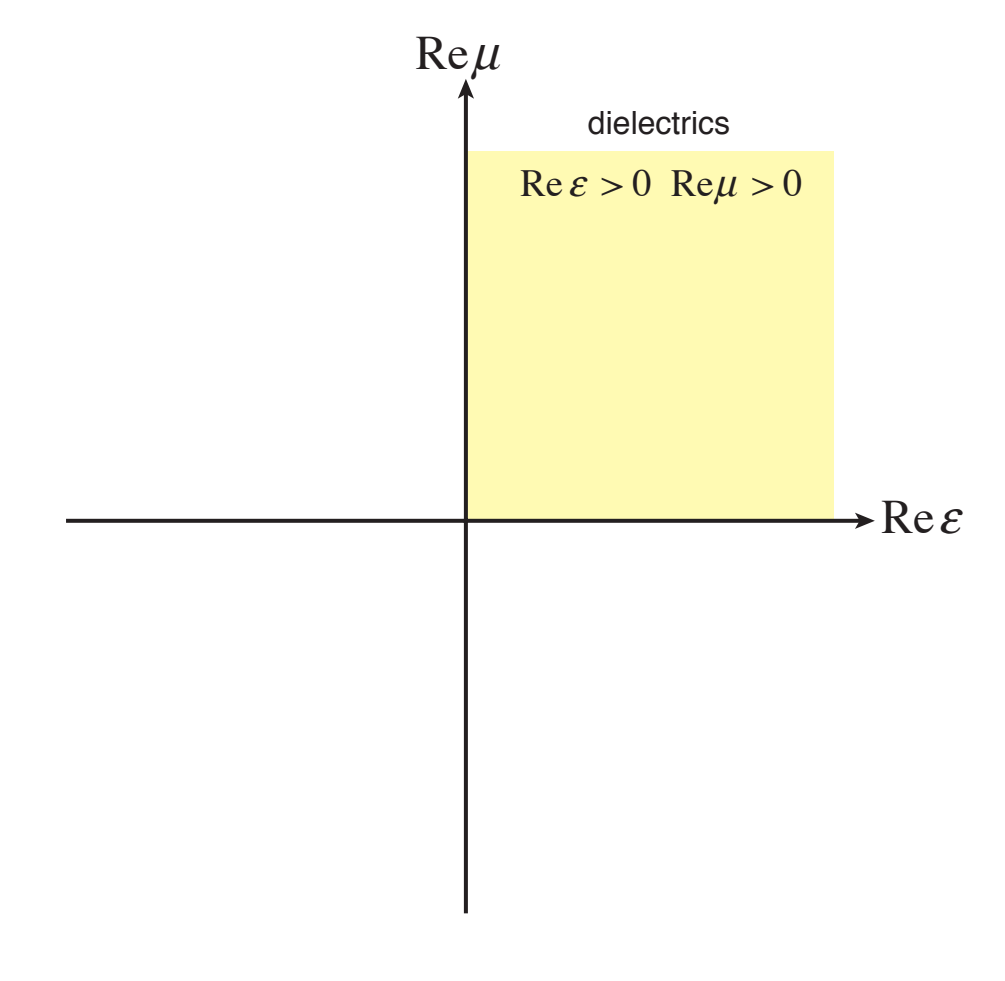

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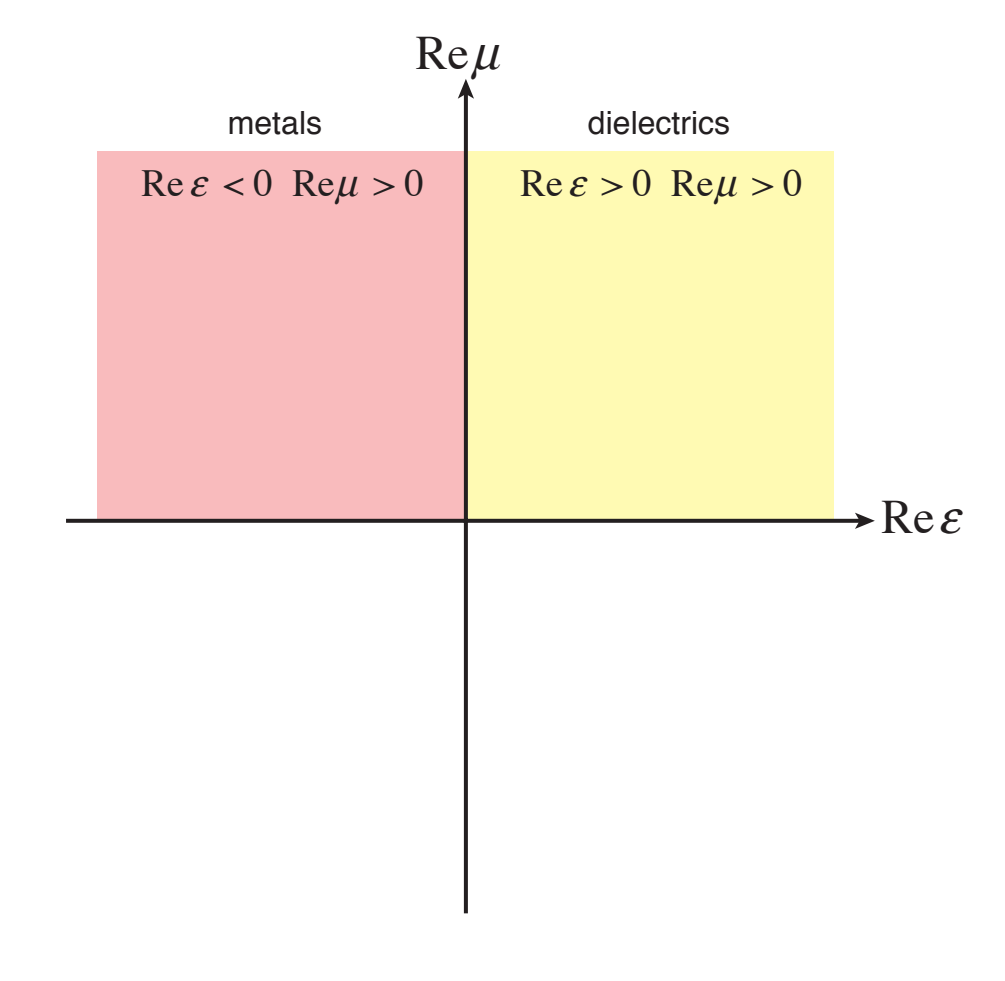

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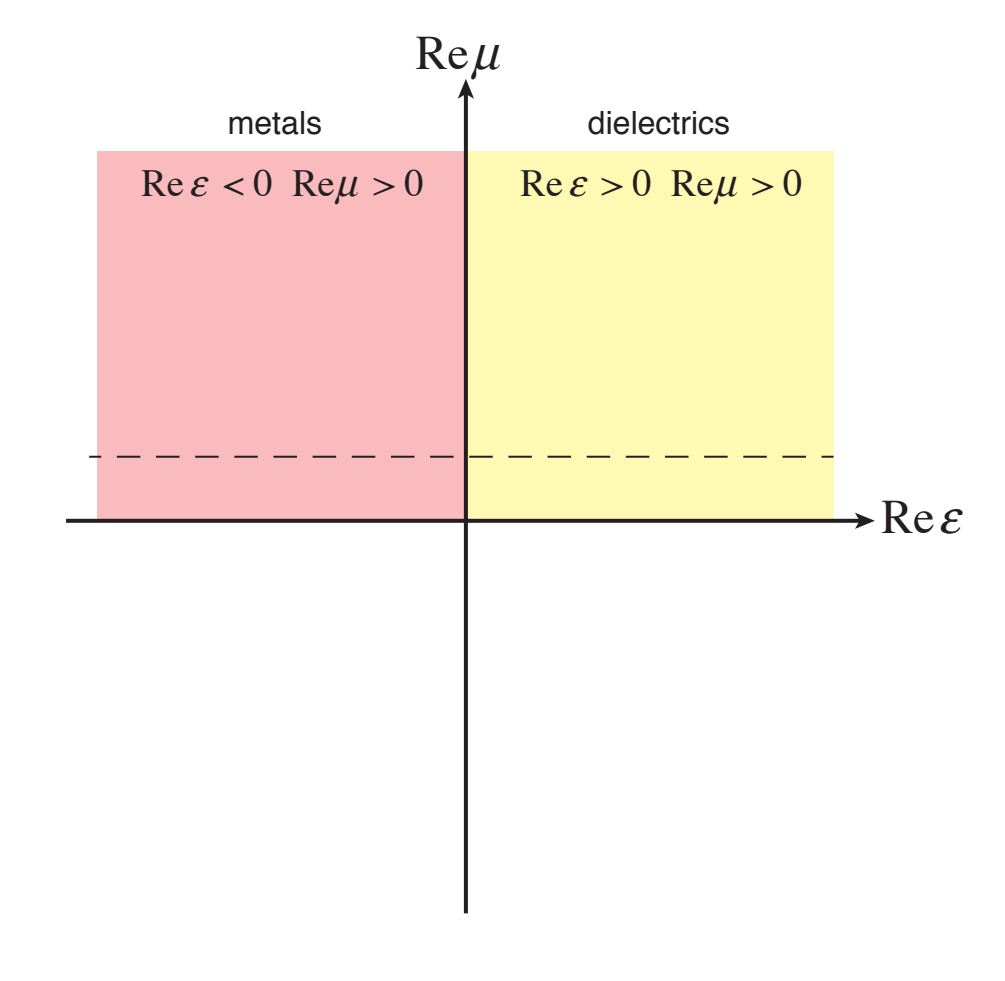

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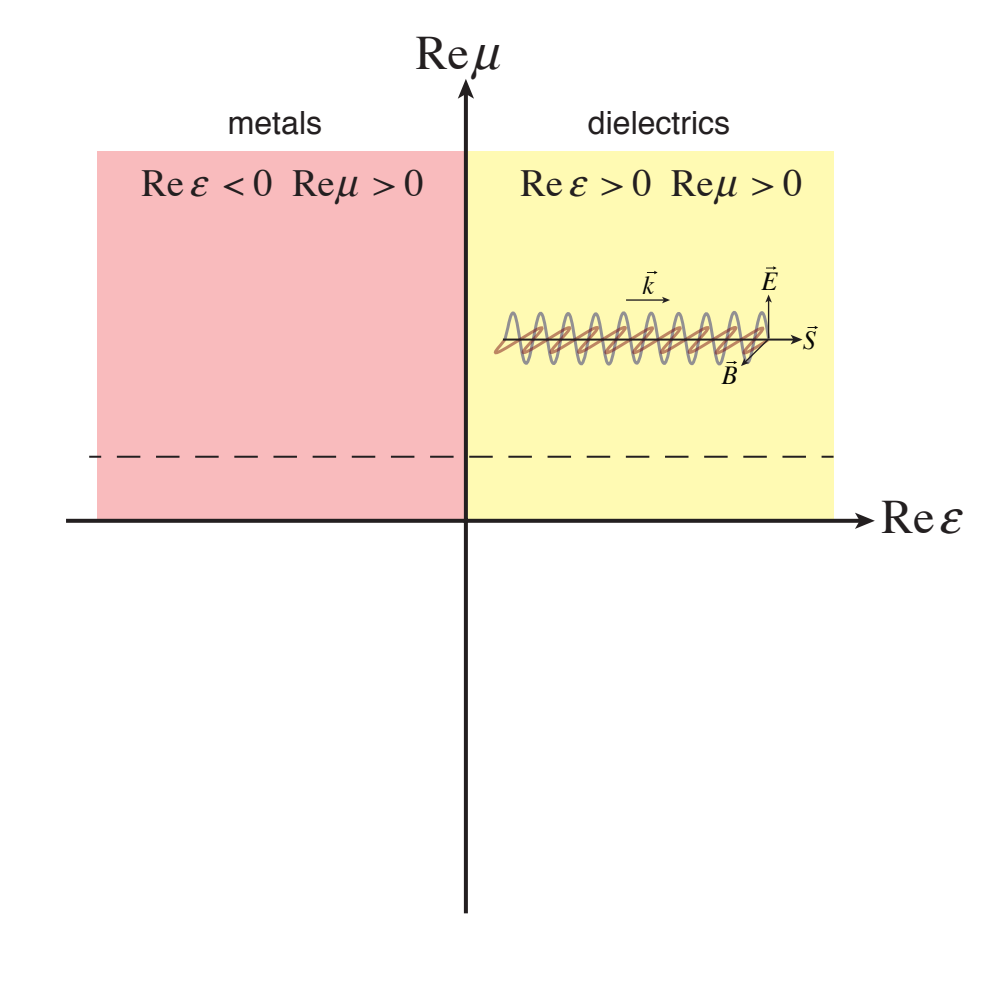

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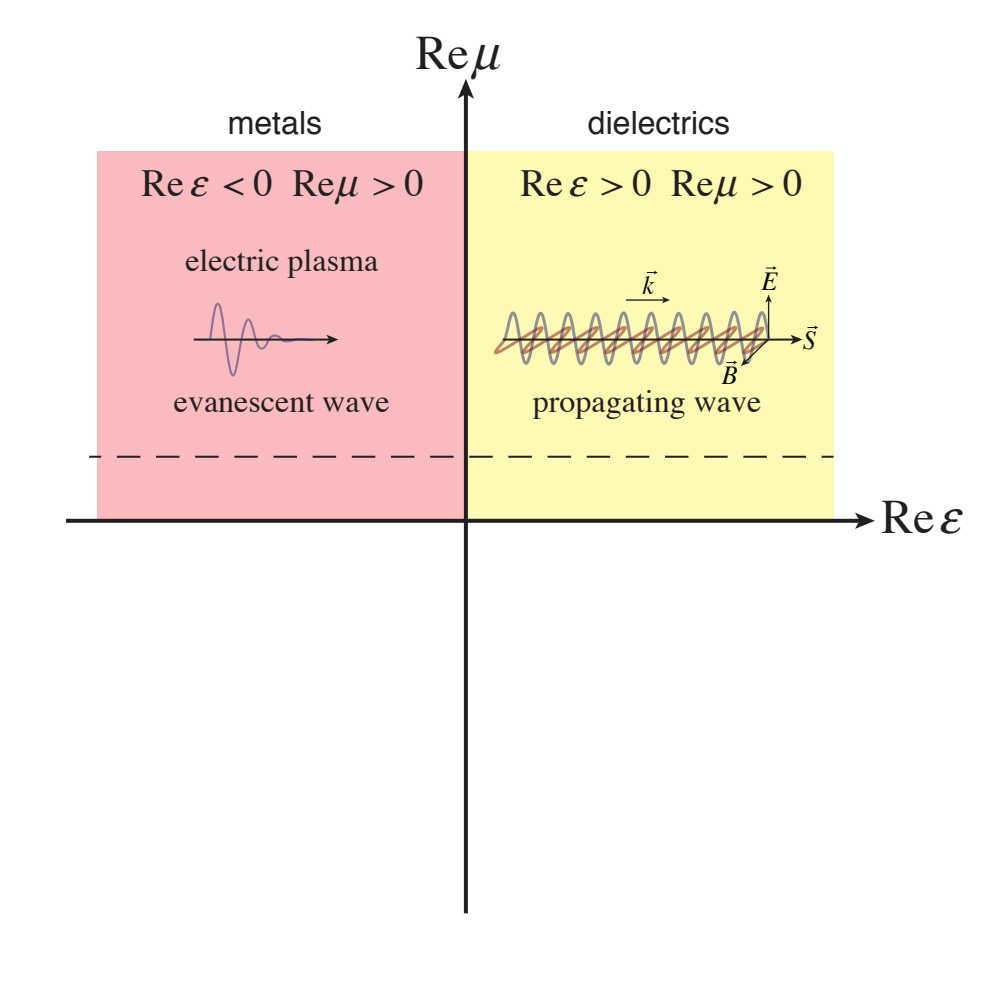

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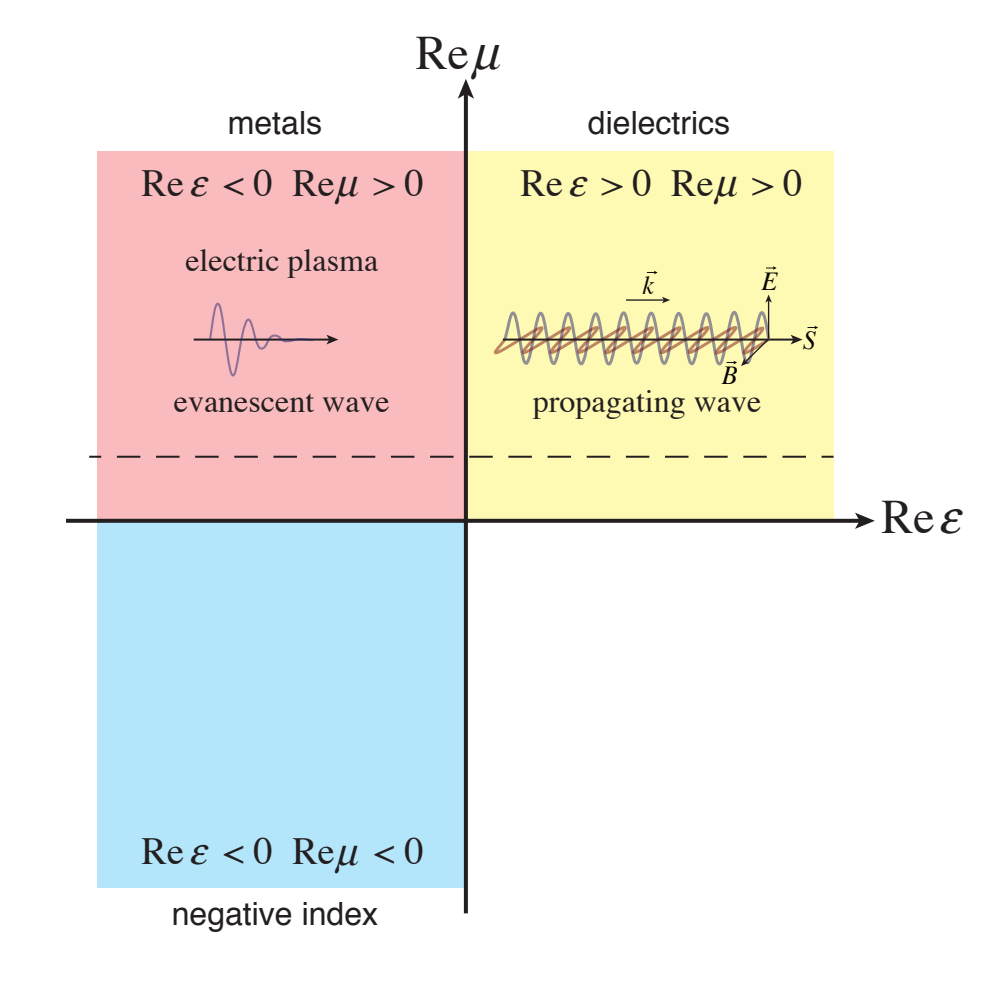

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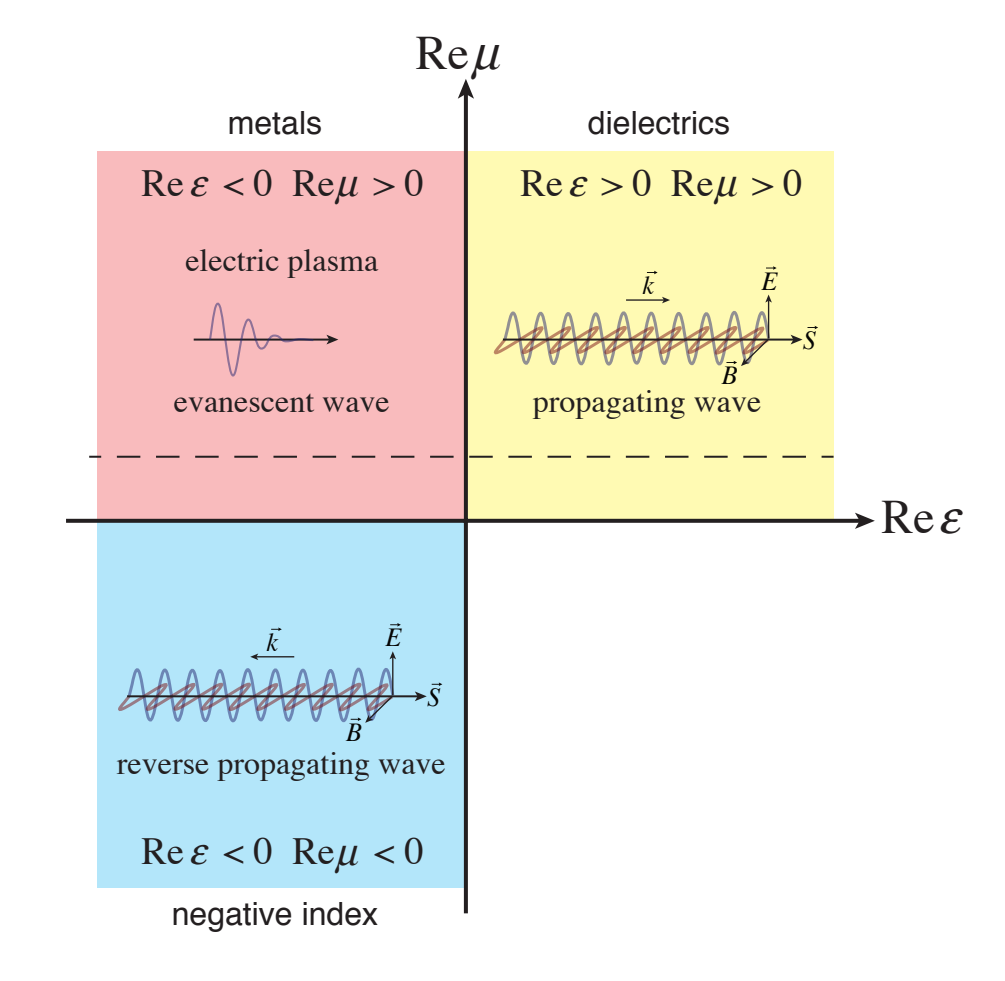

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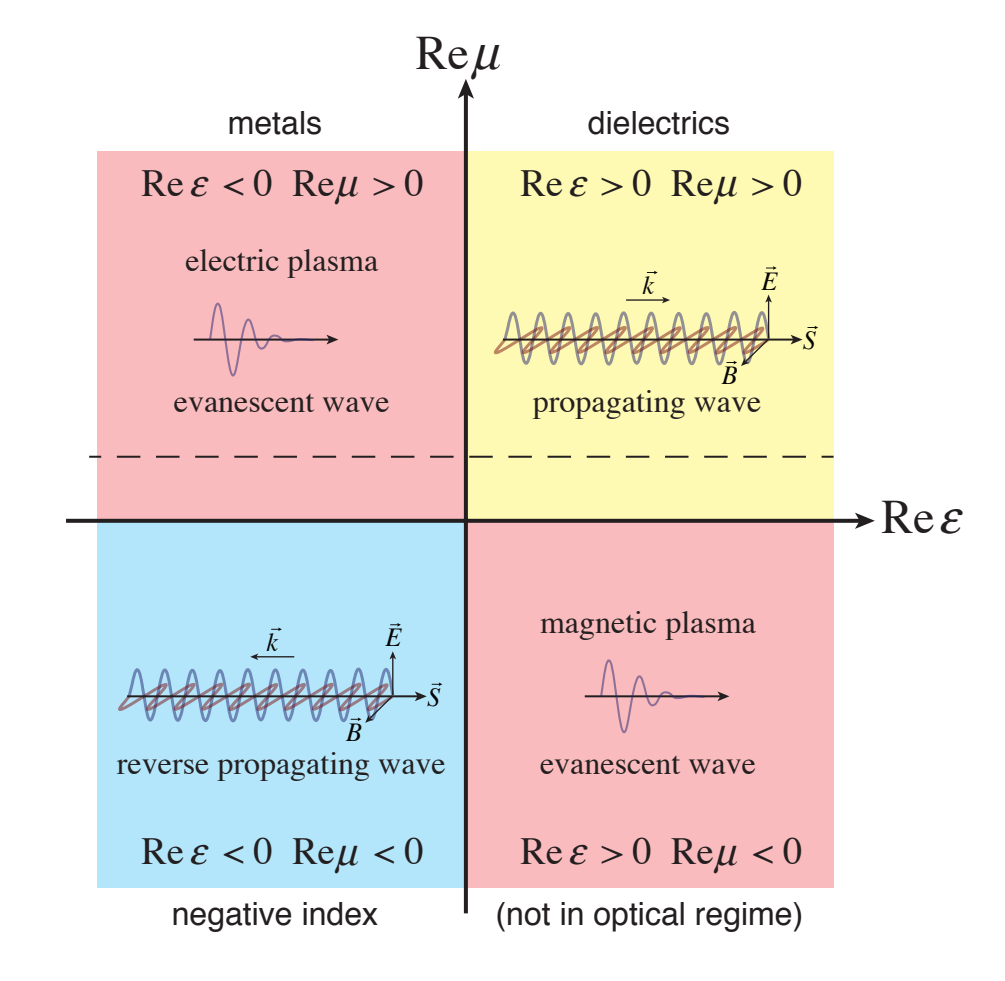

Optical properties of materials

Key points

- optical properties arise from motion of charge
- optical properties depend strongly on driving frequency
- index positive, zero, imaginary, or even negative

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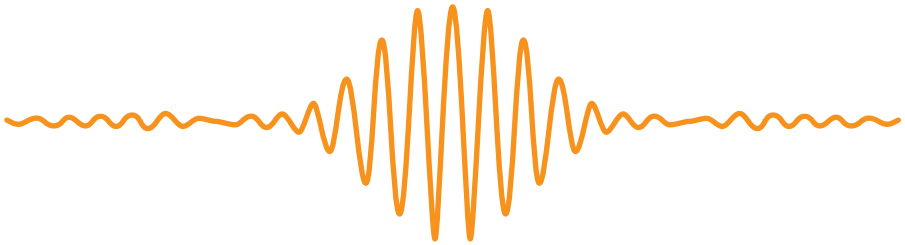
Why did we do all this?

Outline

- optical properties of materials
- **dispersion of pulses**
- nonlinear optics
- waveguiding
- engineering the index

Pulse dispersion



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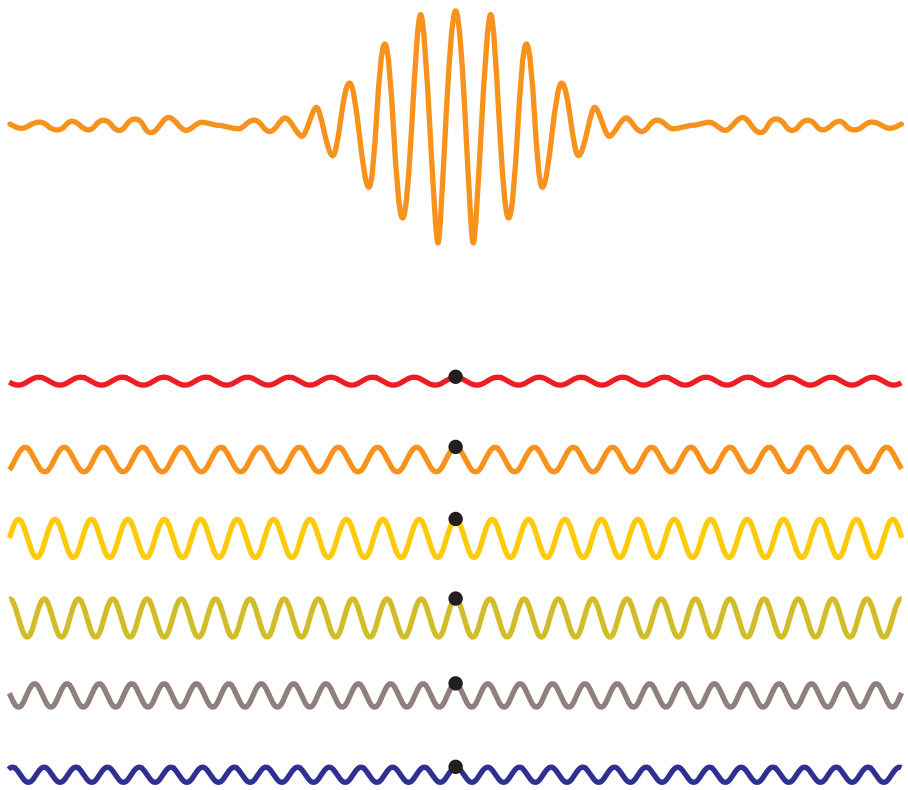
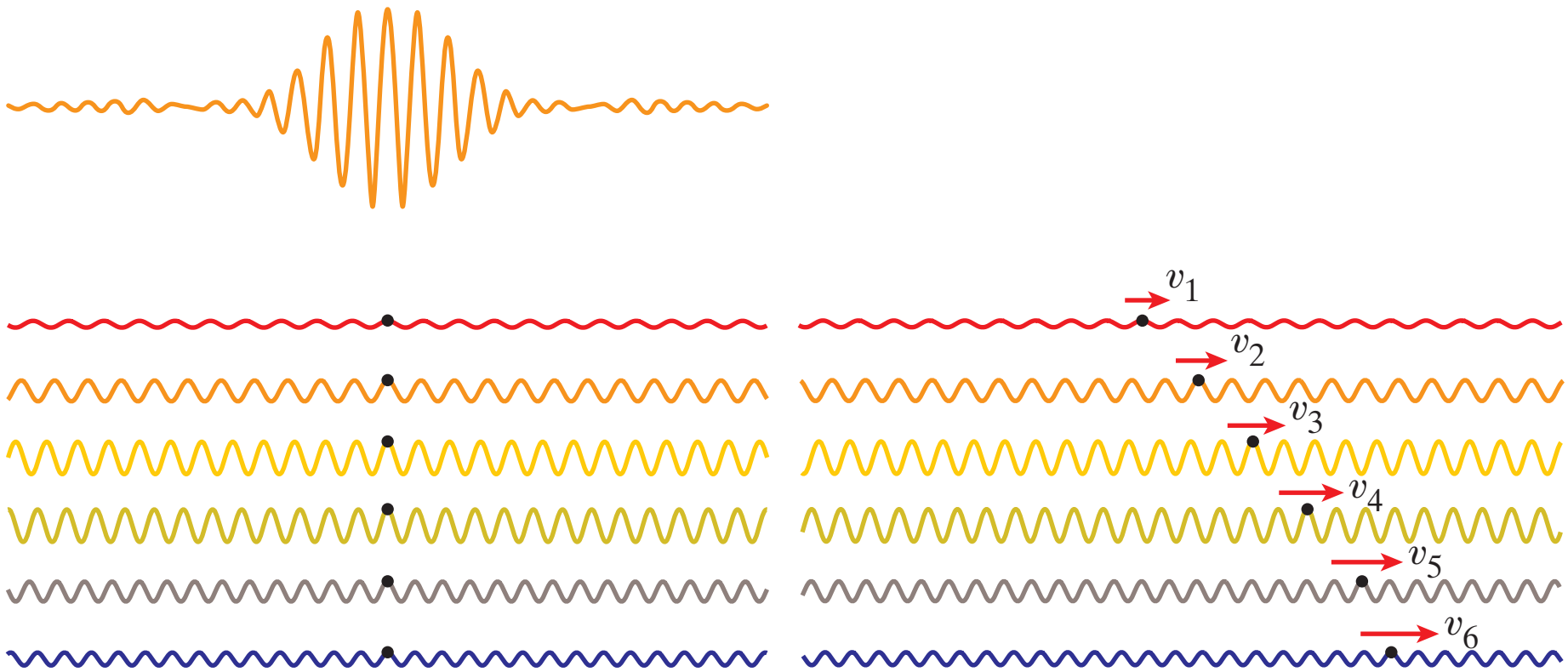
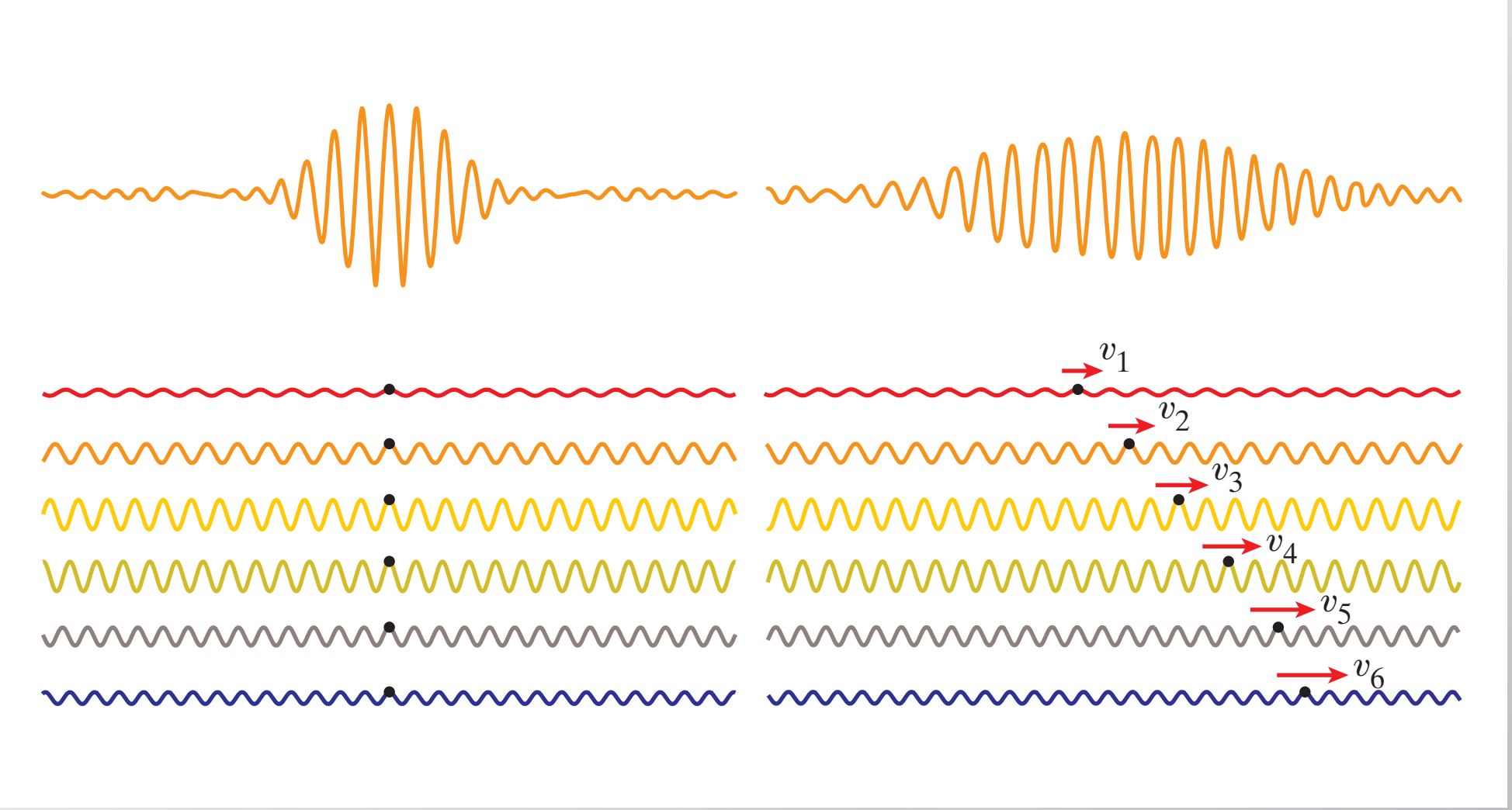


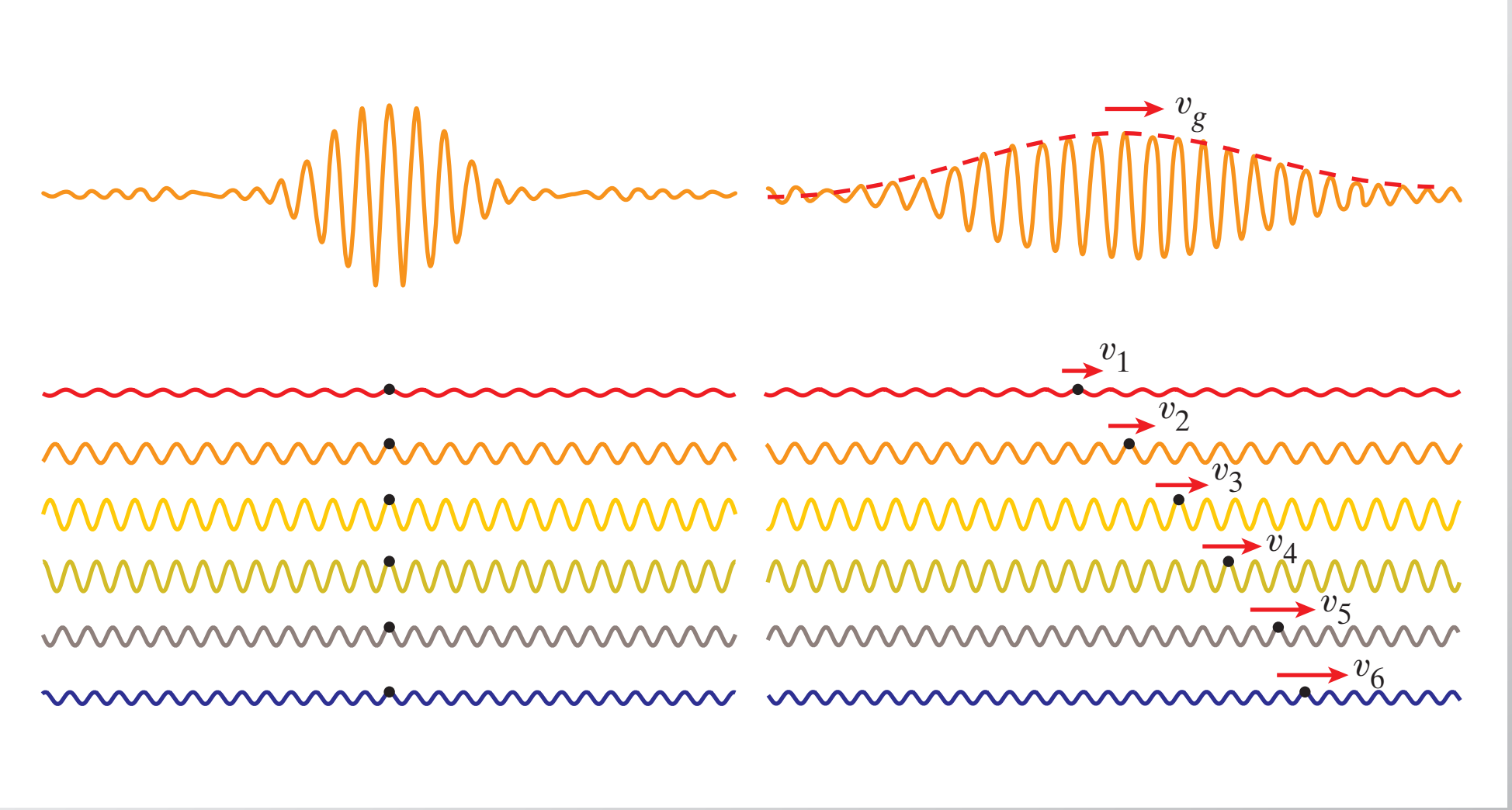
Pulse dispersion



Pulse dispersion



Pulse dispersion



Pulse dispersion

Consider two propagating waves:

$$y_1 = A \sin(k_1 x - \omega_1 t) \quad \textbf{and} \quad y_2 = A \sin(k_2 x - \omega_2 t)$$

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$$v_1 = \frac{\omega_1}{k_1} = f_1 \lambda_1 \quad \textbf{and} \quad v_2 = \frac{\omega_2}{k_2} = f_2 \lambda_2 \,.$$

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Superposition:

$$y = A[\sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t)]$$

Pulse dispersion

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$$y = A[\sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t)]$$

$$\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\alpha + \beta}{2}\right)$$

Pulse dispersion

$$y = 2A \cos \tfrac{1}{2}[(k_1 - k_2)x - (\omega_1 - \omega_2)t] \sin \left[\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t \right]$$

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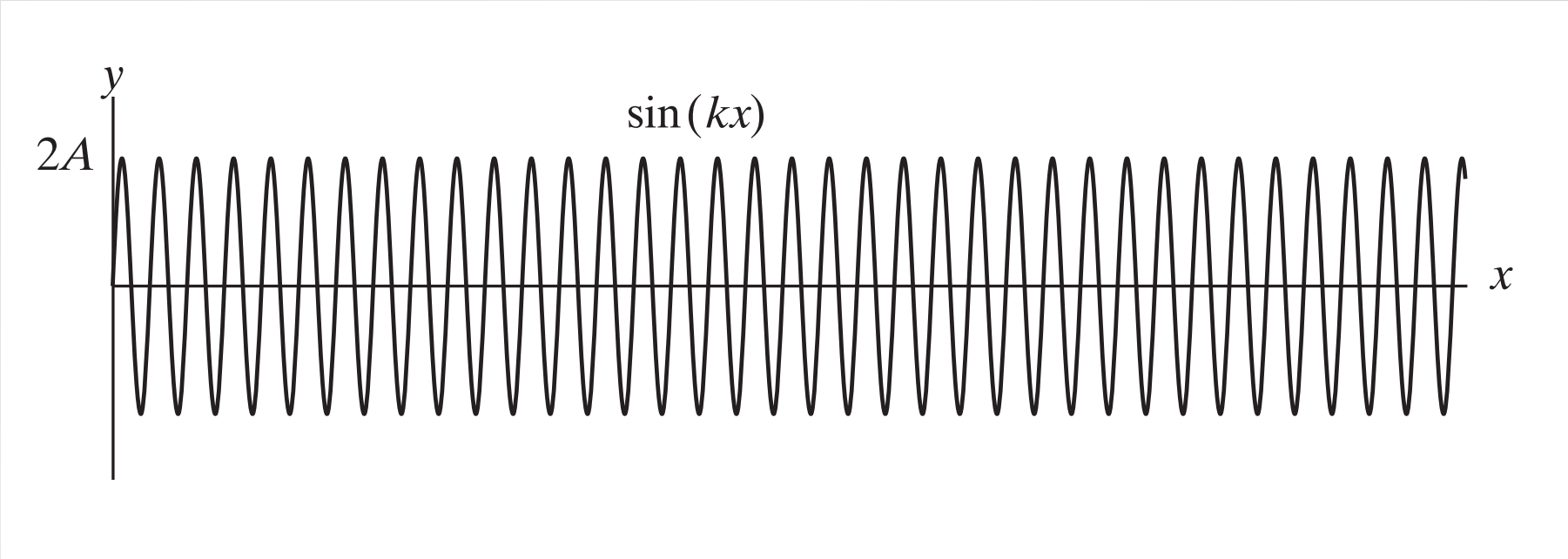
traveling sine wave, with amplitude modulation.

Pulse dispersion

$$y = 2A \cos \tfrac{1}{2}(x\Delta k - t\Delta\omega) \sin(kx - \omega t)$$

At $t = 0$**:**

$$y = 2A \cos \tfrac{1}{2}(x\Delta k) \sin(kx)$$



Pulse dispersion

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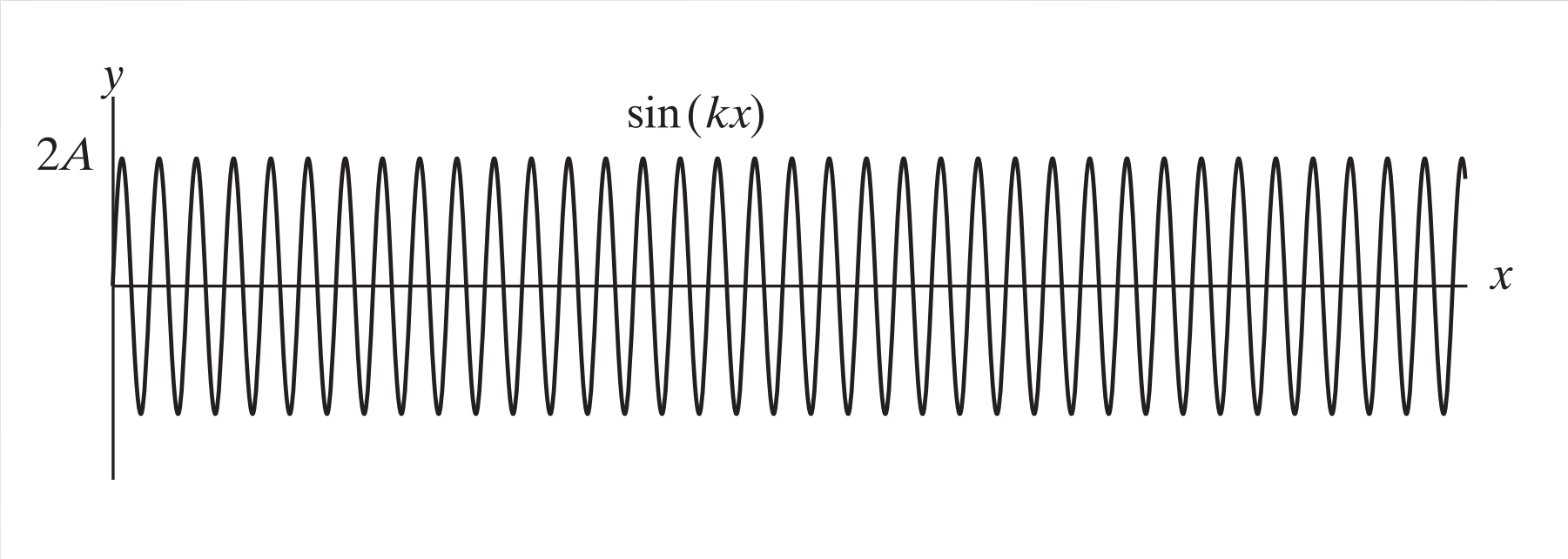
carrier



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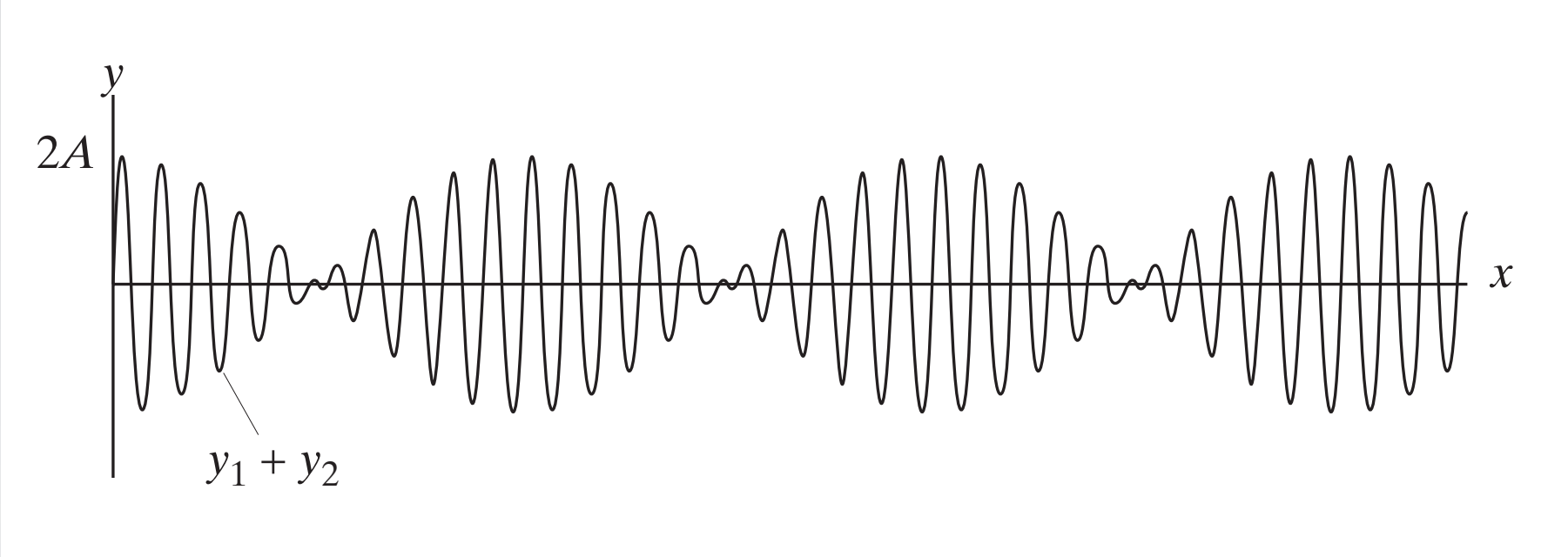

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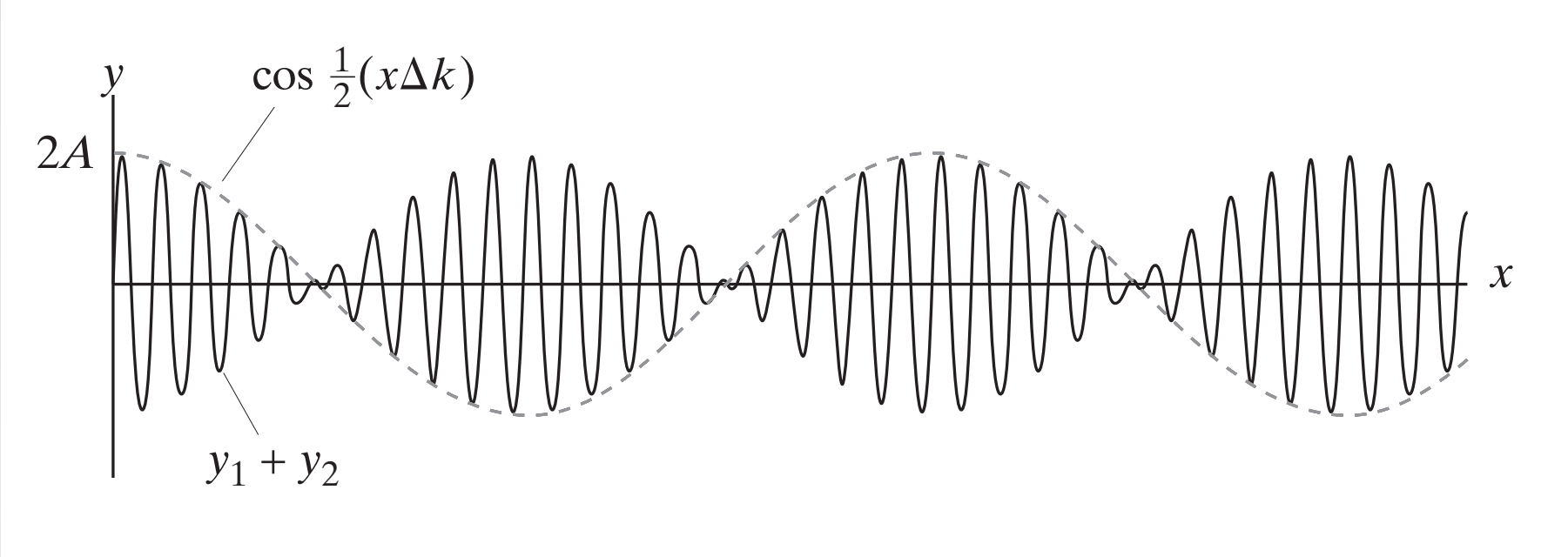
envelope **carrier**



Pulse dispersion

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Pulse dispersion

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speed of carrier

$$v_p = \frac{\omega}{k} = f\lambda$$

Pulse dispersion

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speed of envelope

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Pulse dispersion

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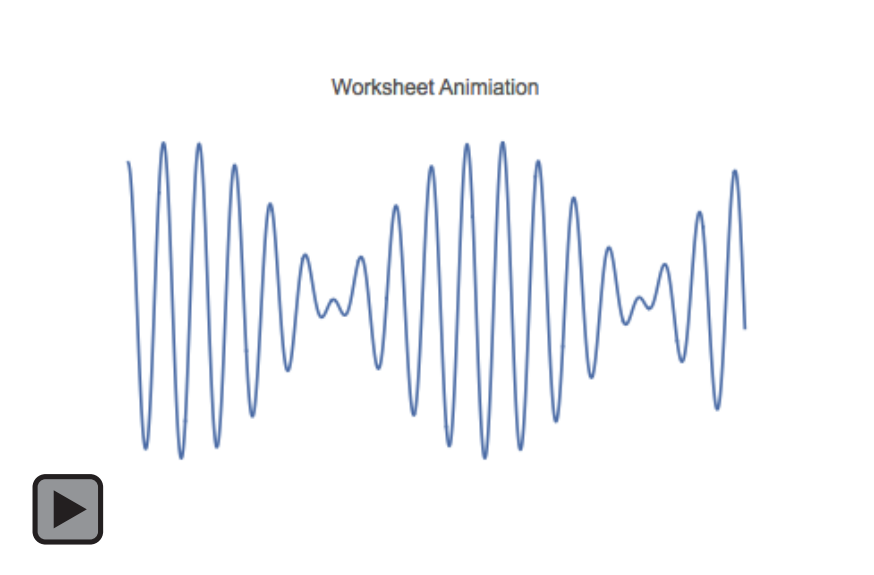
speed of carrier ('phase velocity'):

$$v_p = \frac{\omega}{k} = f\lambda$$

speed of envelope ('group velocity'):

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Pulse dispersion



let's practice a bit!

(please complete worksheet)

Pulse dispersion

For each wave, determine the wavevector k, the frequency ω, and the propagation speed v:

$k_1 =$ 8.0 and $k_2 = \frac{7.2}{0.95} = 7.6 < k_1$

(R) (B)

$\omega_1 =$ 8.0 and $\omega_2 =$ 7.2

$v_1 = \frac{\omega_1}{k_1} = 1.$ and $v_2 = \frac{\omega_2}{k_2} = \frac{7.2}{7.57} = 0.95$

Does the red get ahead of the blue or the other way around? Why?

Is the dispersion in the medium through which these waves propagate normal or anomalous? Why?

Pulse dispersion

What is the phase velocity of the superposition of y_1 and y_2?

$$v_p = \frac{(\omega_1 + \omega_2)/2}{(k_1 + k_2)/2} = \frac{7.6}{7.8} = 0.98$$

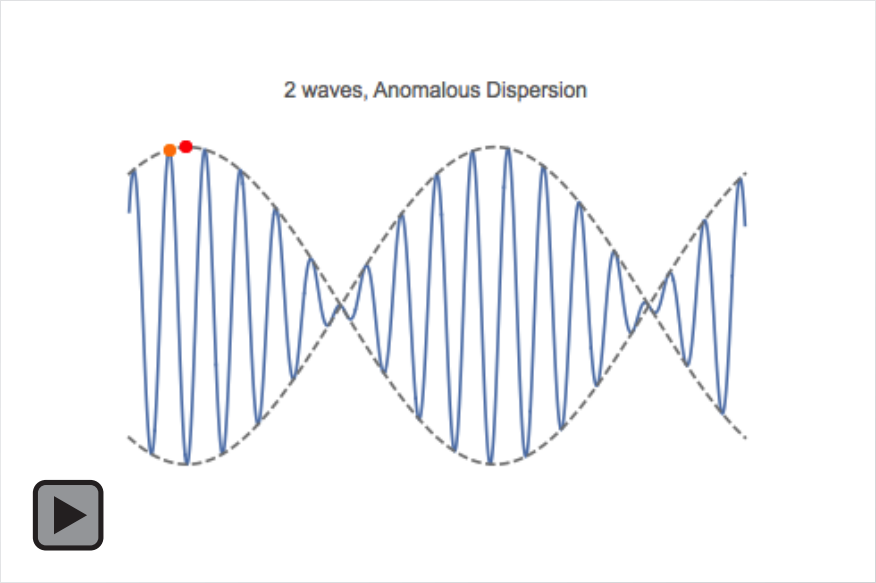
What is the group velocity of the superposition of y_1 and y_2?

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{0.8}{0.4} = 2$$

Do the crests of the carrier wave travel forward or backward through the envelope? Does your prediction agree with your observation?

Pulse dispersion

2 sine waves, anomalous dispersion

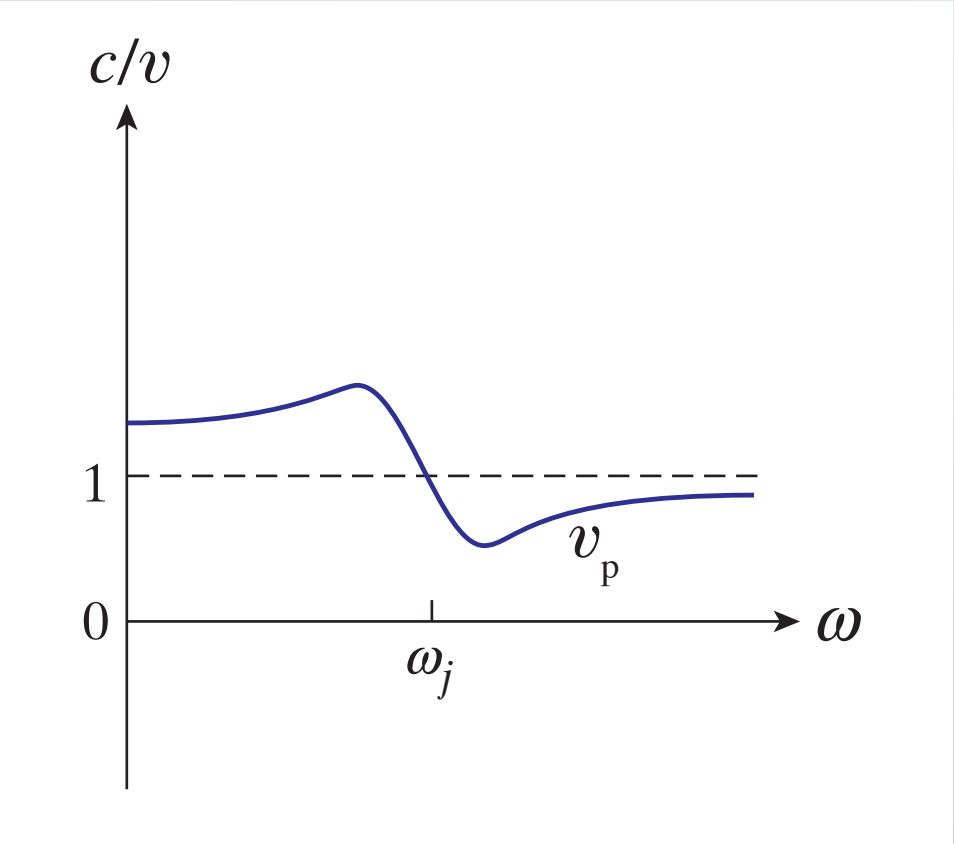


Pulse dispersion

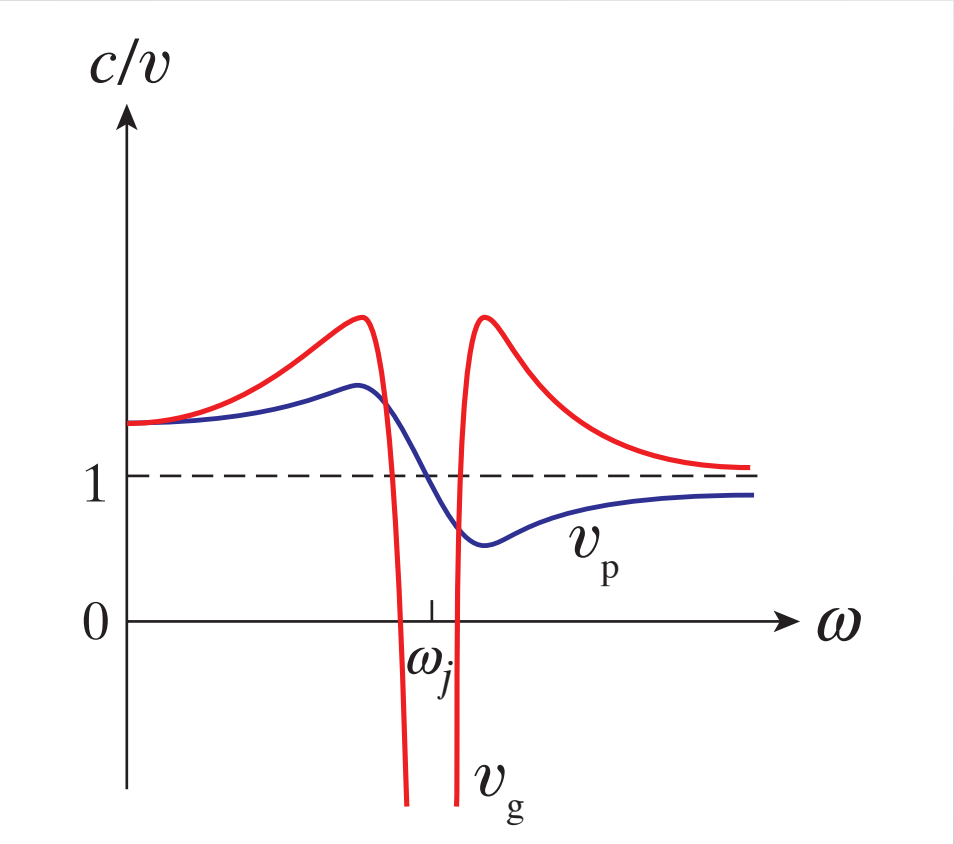
Q: In the previous example, if we take $v = 1$ to mean c, then v_g is "superluminal". Is this possible?

1. no, neither v_p nor v_g can be larger than c.
2. no, only v_p can be larger than c.
3. yes, v_g can be larger than c (but not v_p).
4. yes, both v_p and v_g can be larger than c.

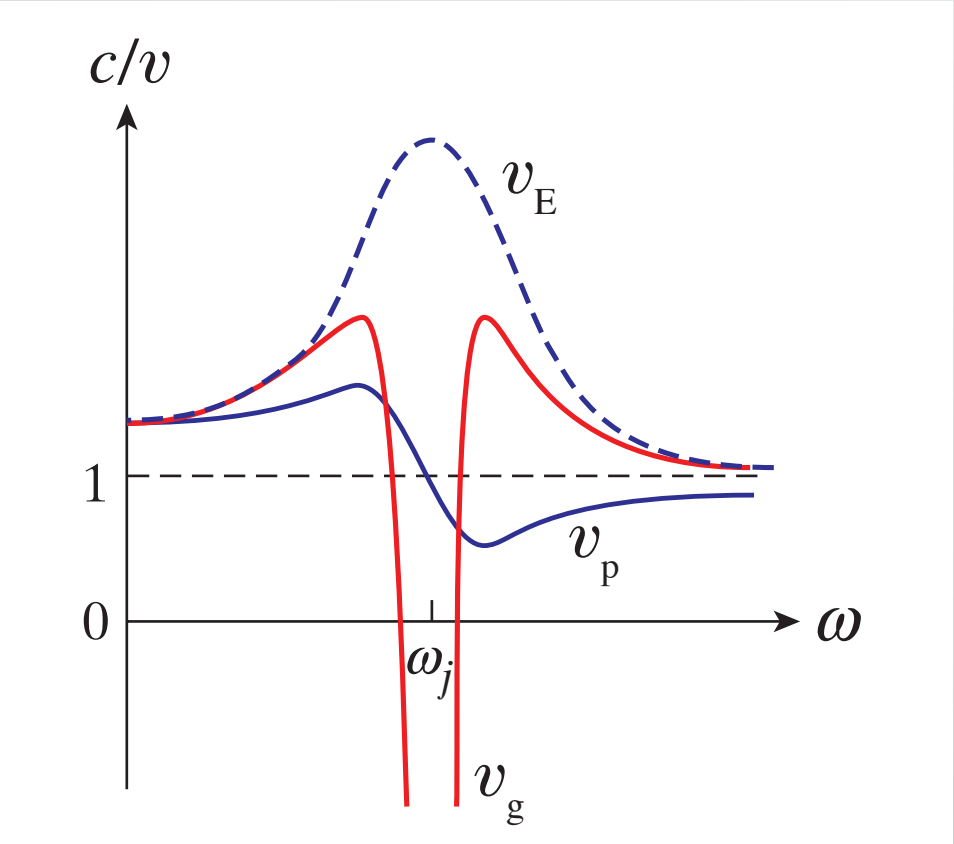
Pulse dispersion



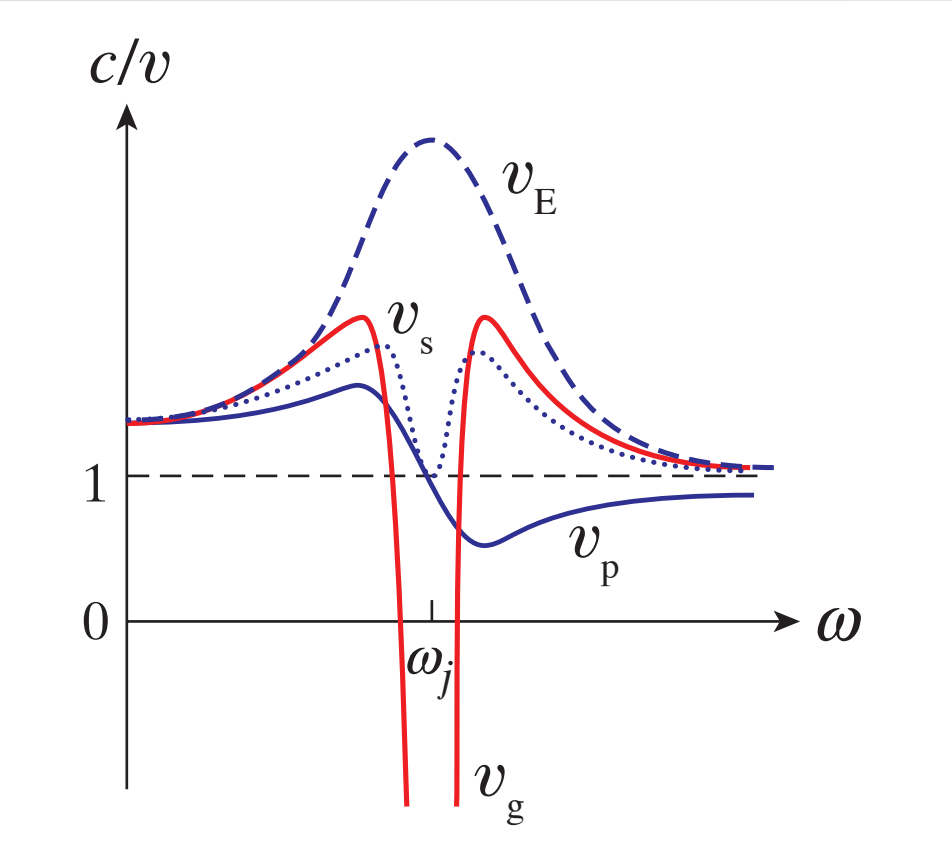
Pulse dispersion



Pulse dispersion



Pulse dispersion



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Pulse dispersion

$$y = 2A \cos \tfrac{1}{2}(x\Delta k - t\Delta\omega) \sin(kx - \omega t)$$

If no dispersion

$$v_p = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$$

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$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\omega_1/k_1 - \omega_2/k_1}{1 - k_2/k_1} = \frac{v_p - \omega_2/k_1}{1 - k_2/k_1}$$

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$$v_g = \frac{v_p - \omega_2/k_2(k_2/k_1)}{1 - k_2/k_1} = v_p \frac{1 - k_2/k_1}{1 - k_2/k_1} = v_p$$

Pulse dispersion

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If no dispersion $$v_p = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$$

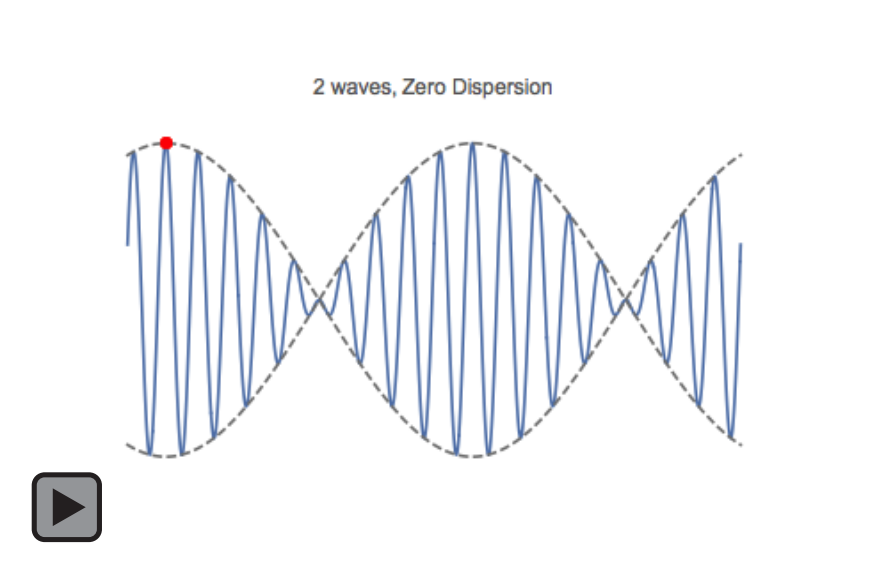
group and phase velocities are the same:

$$v_g = v_p$$

and so the envelope and carrier travel together

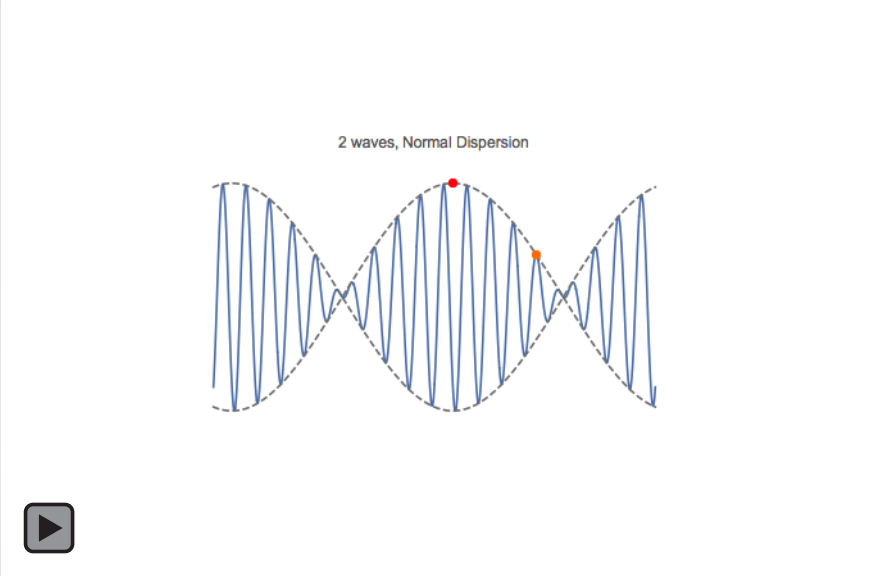
Pulse dispersion

2 sine waves, no dispersion



Pulse dispersion

2 sine waves, normal dispersion



Pulse dispersion

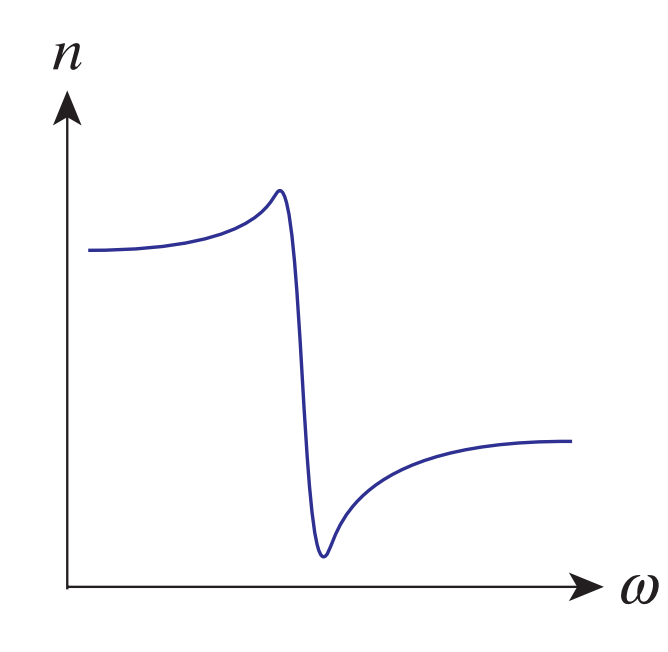
$$y = 2A \cos \tfrac{1}{2}(x\Delta k - t\Delta\omega) \sin(kx - \omega t)$$

Types of dispersion:

$\frac{dn}{d\omega} > 0$ **normal dispersion**

$\frac{dn}{d\omega} = 0$ **no dispersion**

$\frac{dn}{d\omega} < 0$ **anomalous dispersion**



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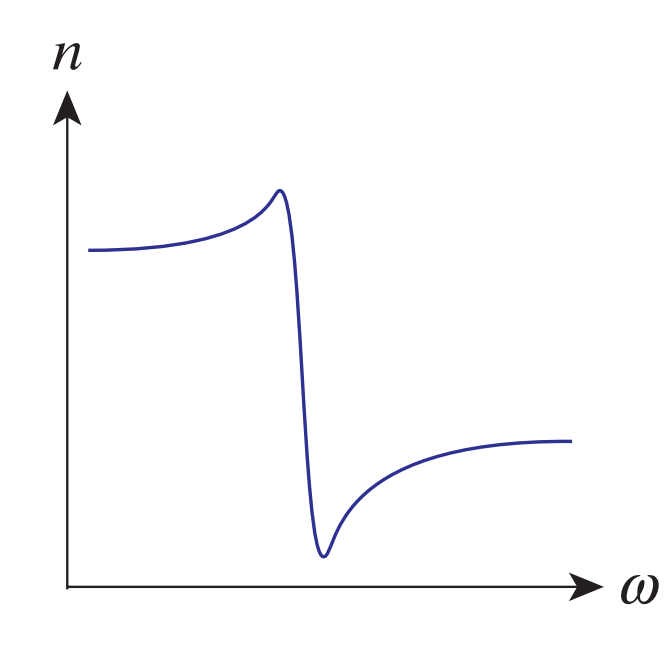
Types of dispersion:

$\frac{dn}{d\omega} > 0$ **normal dispersion** $v_g < v_p$

$\frac{dn}{d\omega} = 0$ **no dispersion** $v_g = v_p$

$\frac{dn}{d\omega} < 0$ **anomalous dispersion** $v_g > v_p$



Pulse dispersion

What is the relationship between phase and group velocities?

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We have: $v_g = \dfrac{d\omega}{dk}$ **and** $k = nk_o$

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and so $\dfrac{1}{v_g} = \dfrac{dk}{d\omega} = k_o \dfrac{dn}{d\omega} + n\dfrac{dk_o}{d\omega} =$

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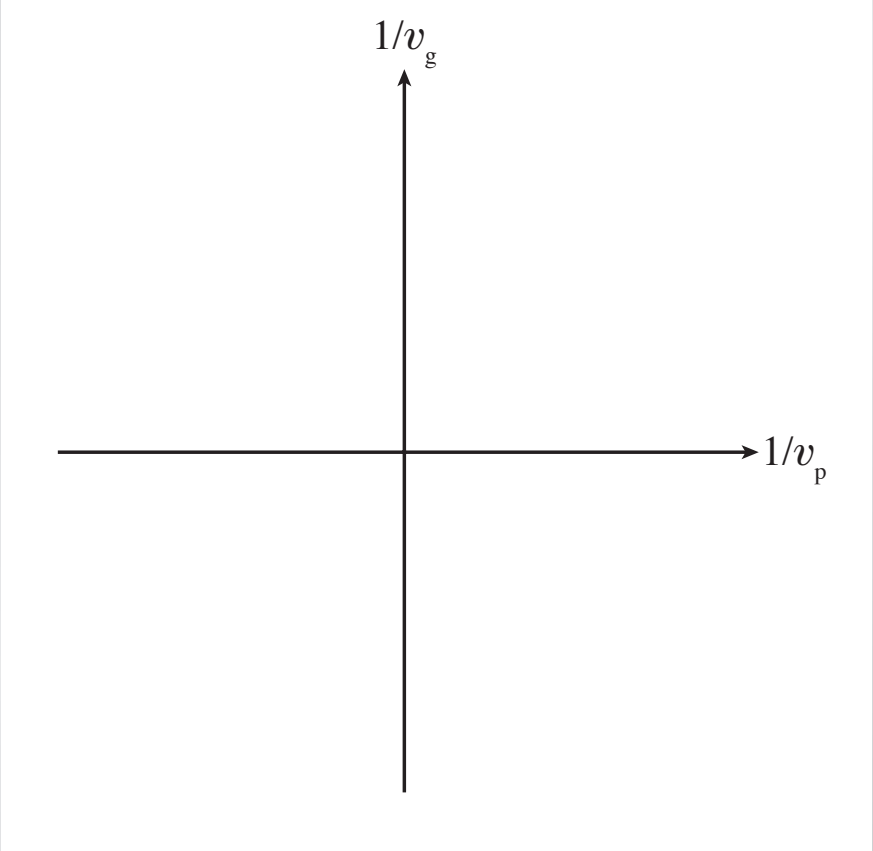
$$= k_o \frac{dn}{d\omega} + \frac{n}{c} = k_o \frac{dn}{d\omega} + \frac{1}{v_p}$$

or

$$\frac{1}{v_g} = k_o \frac{dn}{d\omega} + \frac{1}{v_p}$$

Pulse dispersion

relationship between phase and group velocities



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Pulse dispersion

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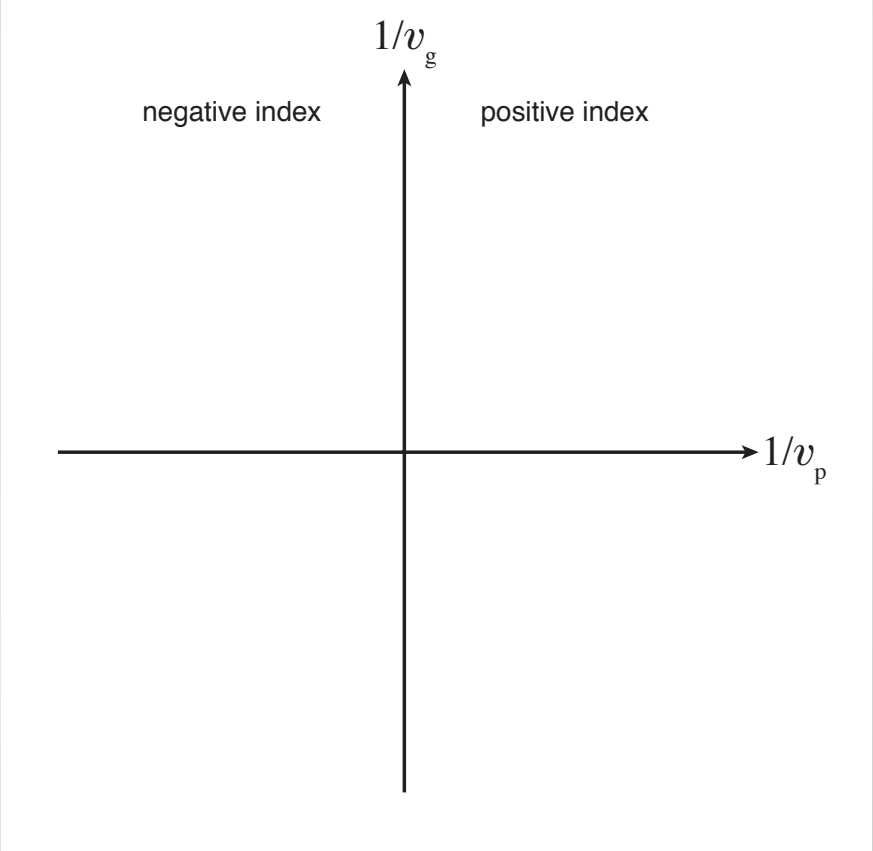

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Pulse dispersion

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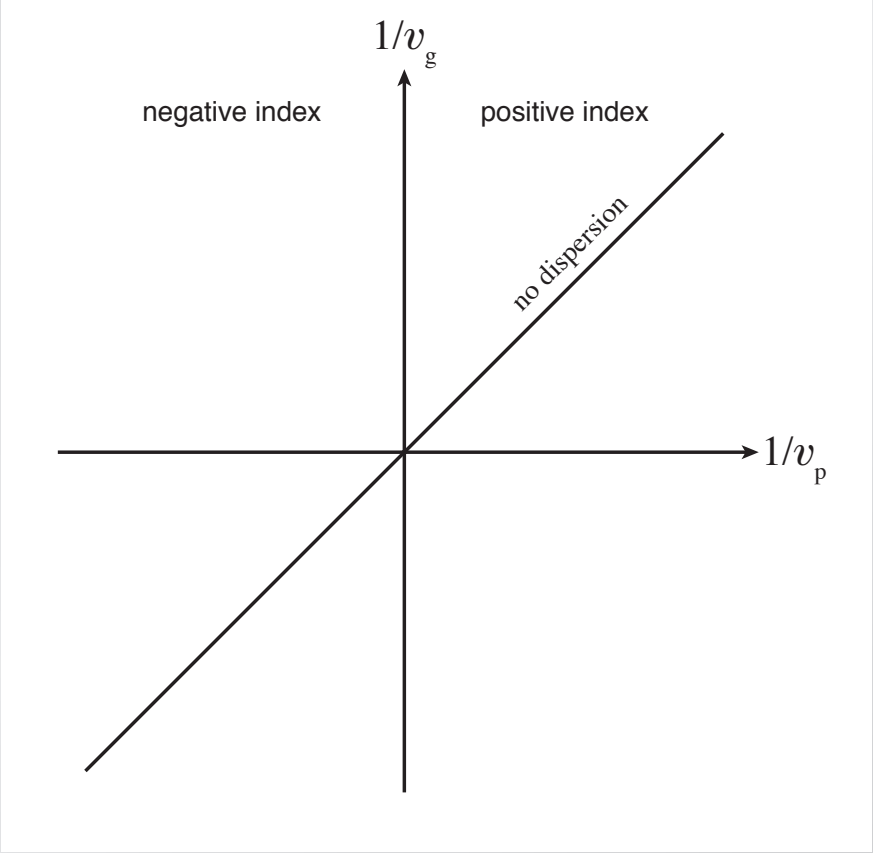

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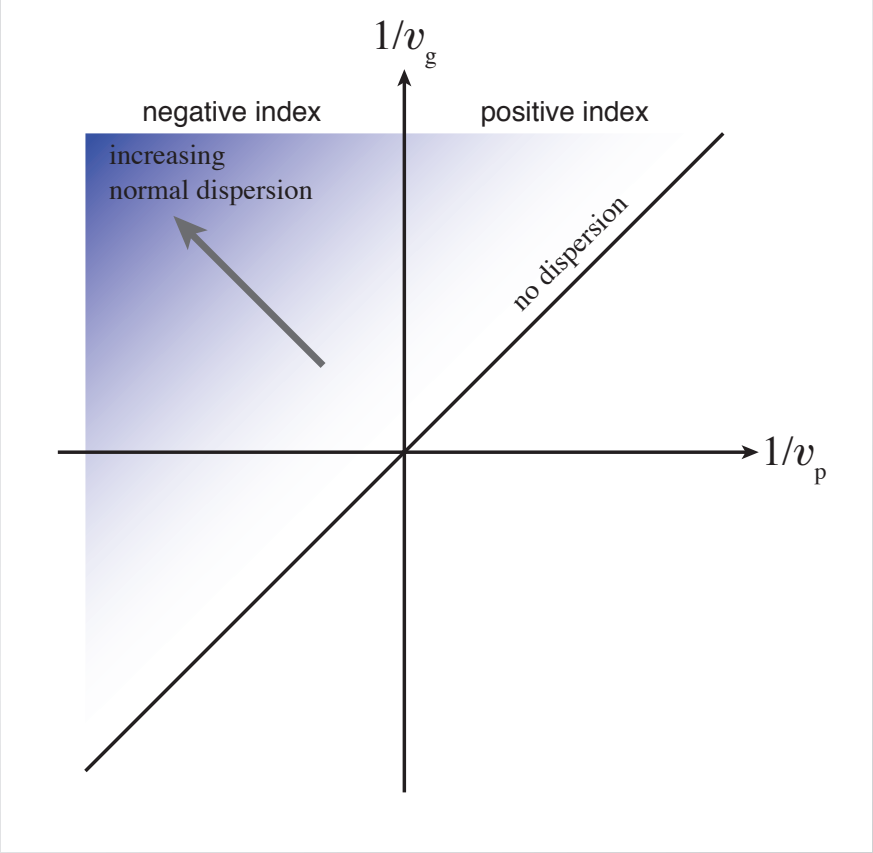
Pulse dispersion

relationship between phase and group velocities



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Pulse dispersion

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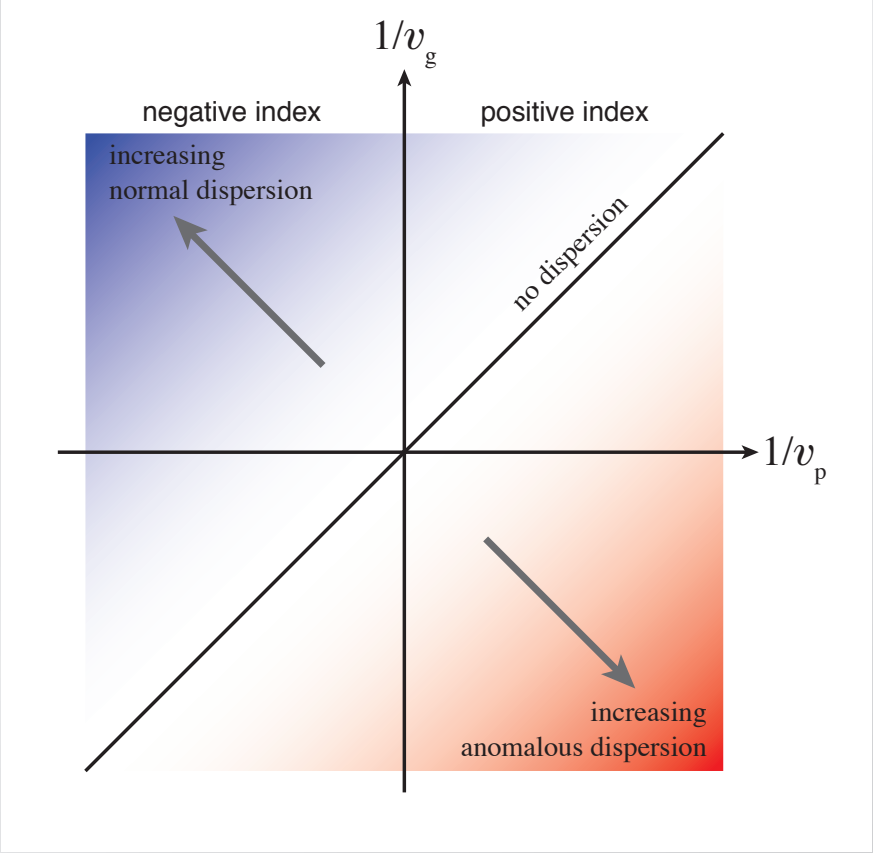

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Pulse dispersion

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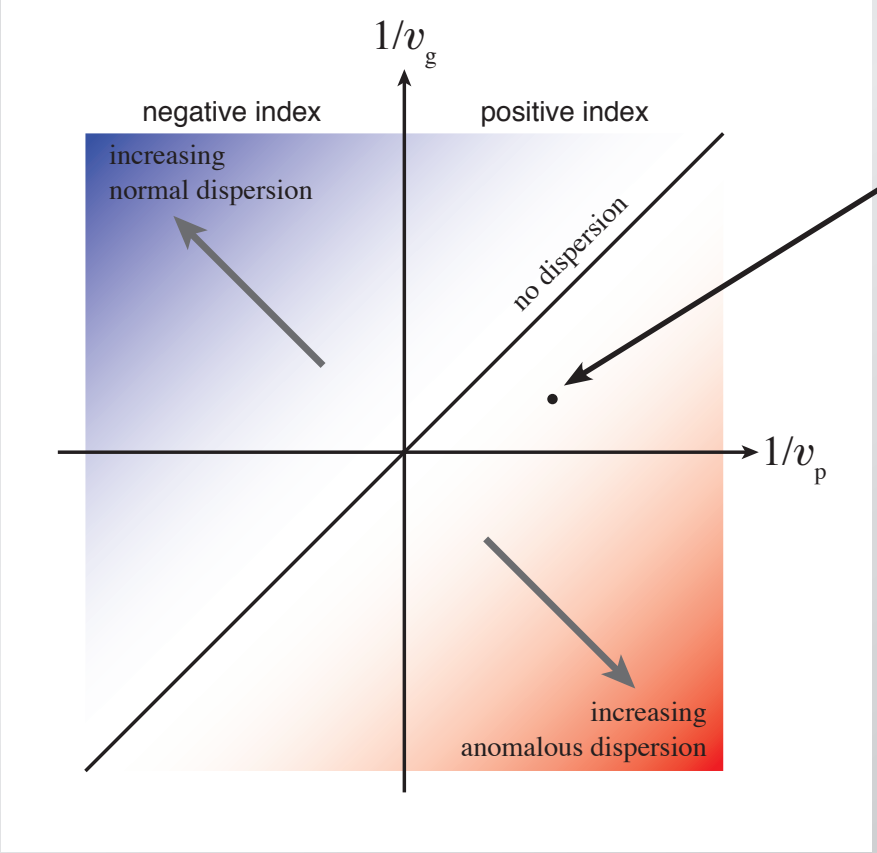


worksheet example

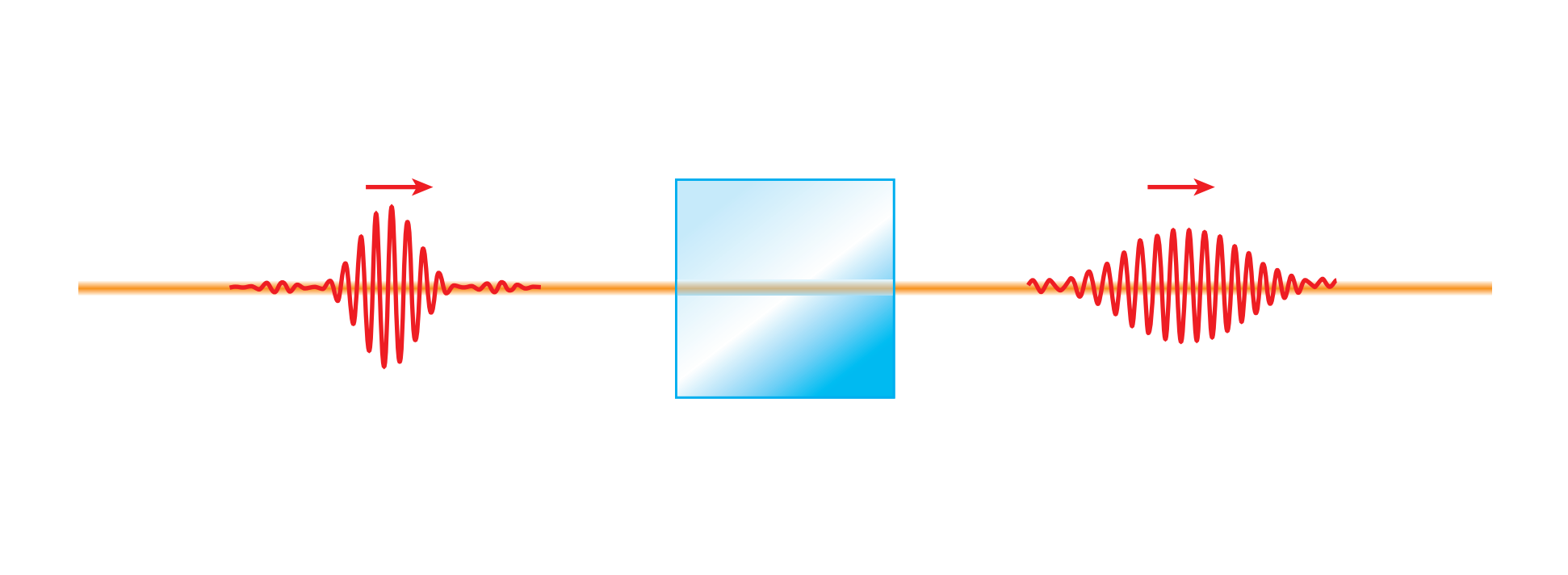
$$\frac{1}{v_g} = k_o \frac{dn}{d\omega} + \frac{1}{v_p}$$

Pulse dispersion

medium causes pulse to stretch

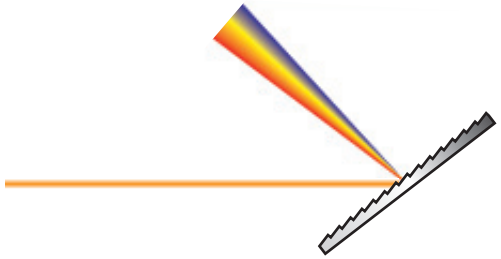
Pulse dispersion

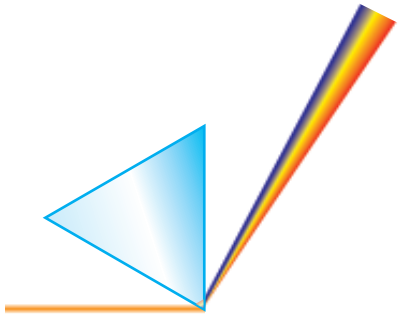
medium causes pulse to stretch



compensate by rearranging spectral components!

Pulse dispersion compensation





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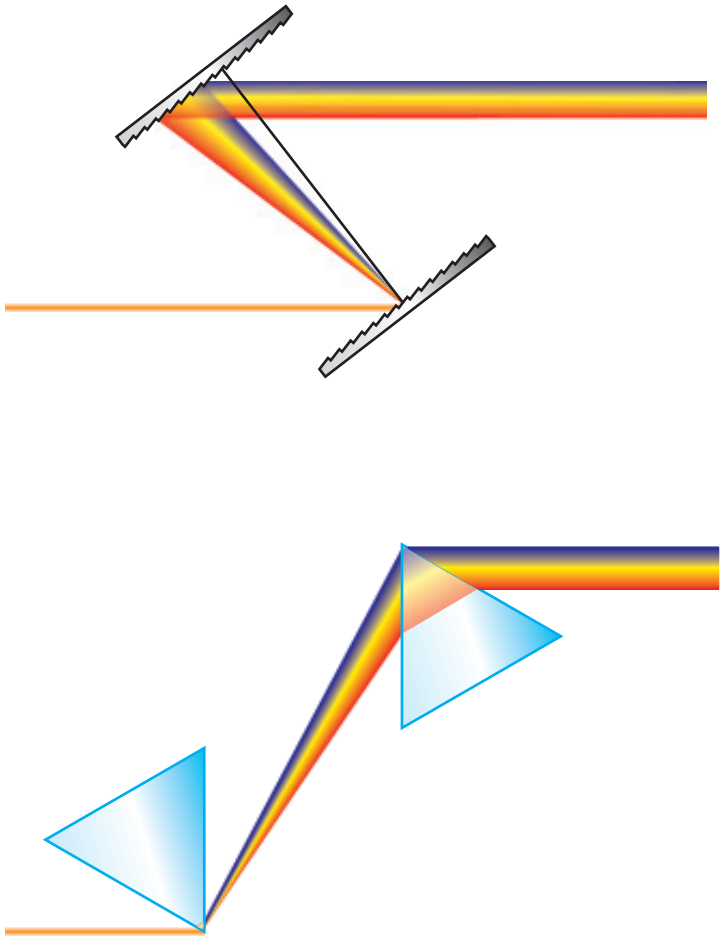

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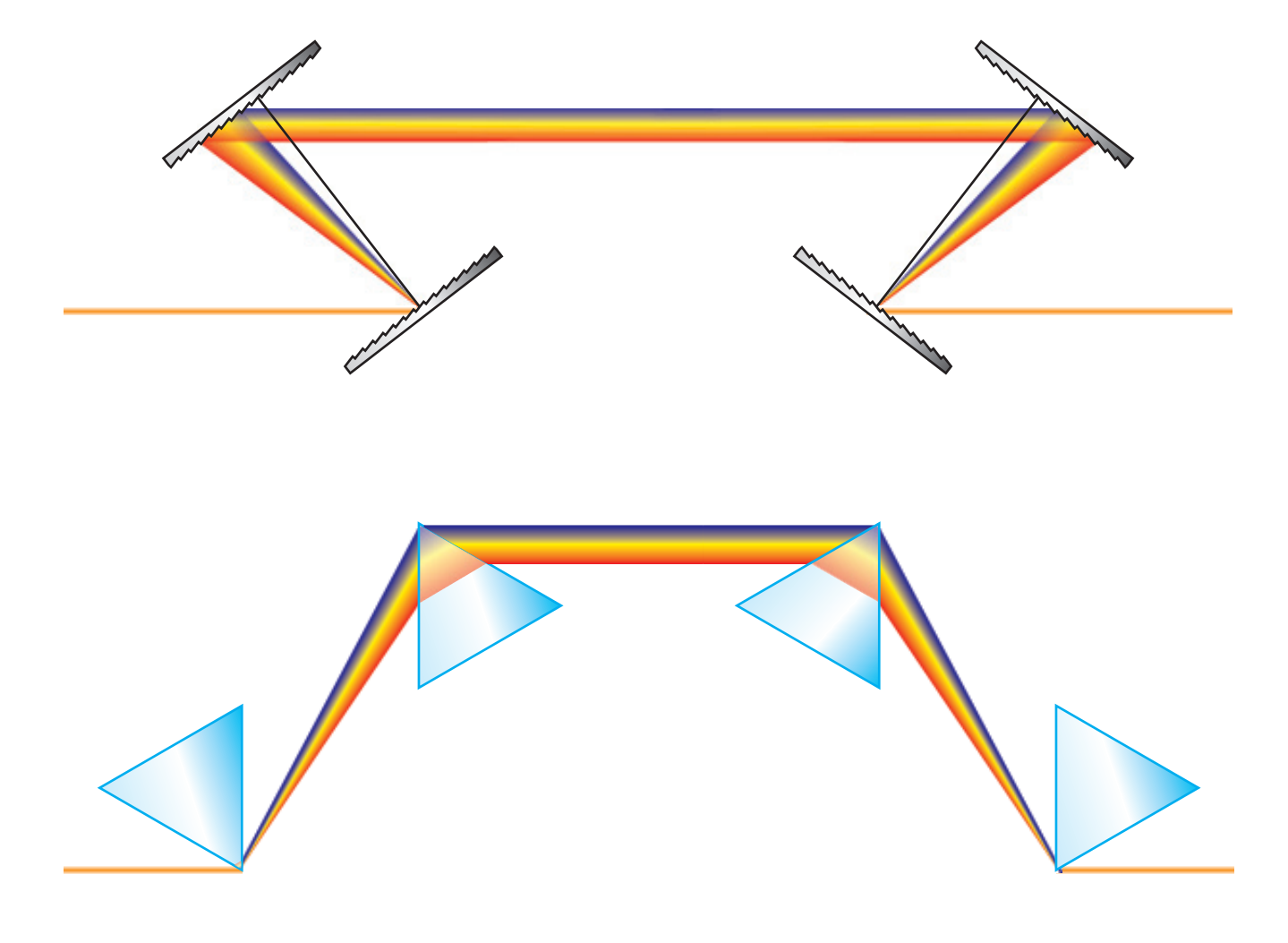
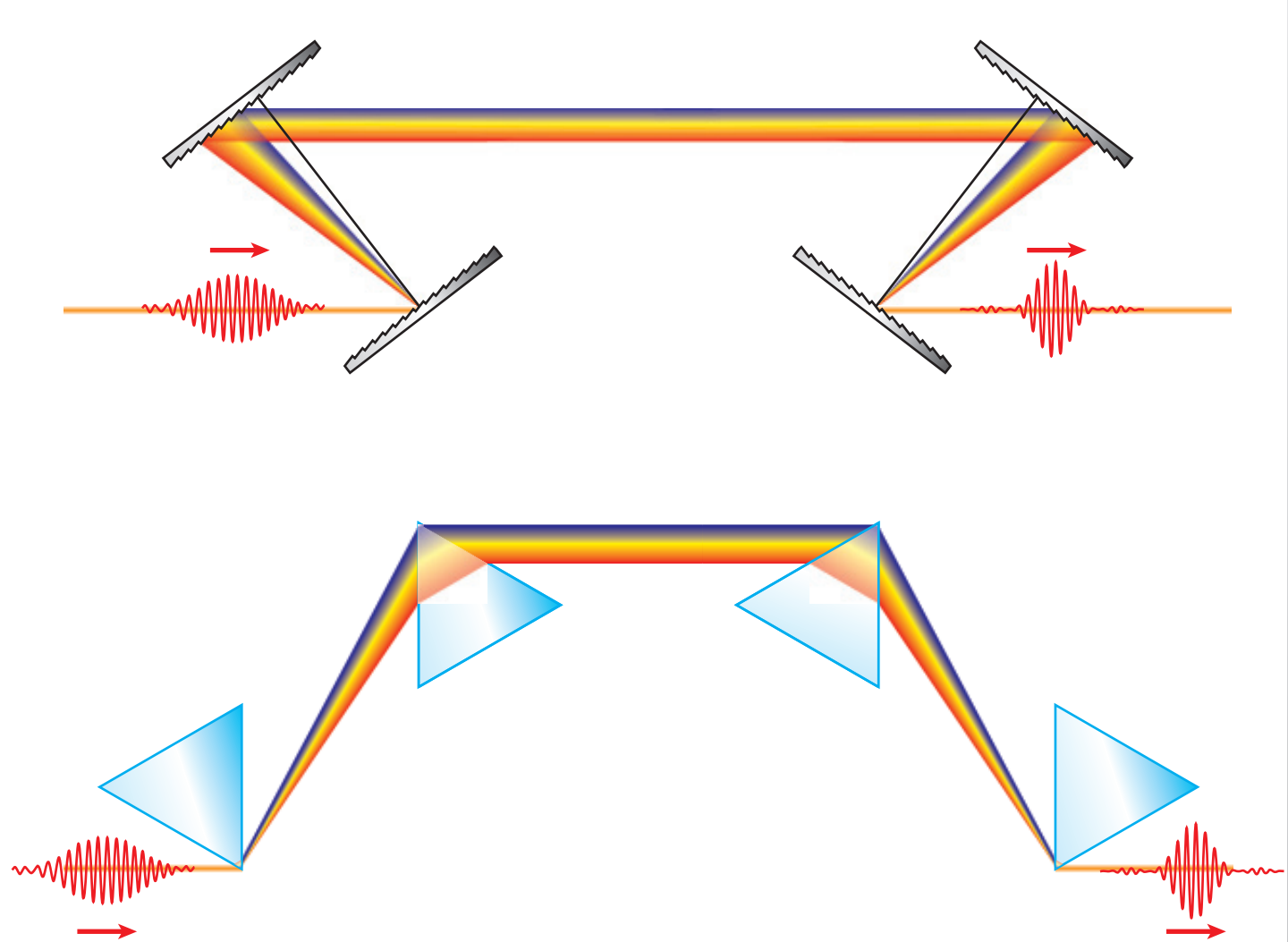


Pulse dispersion compensation



Pulse dispersion compensation

How do these arrangements work?

Pulse dispersion compensation

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(please complete worksheet)

Pulse dispersion compensation

Does path length difference compensate?

Pulse dispersion compensation

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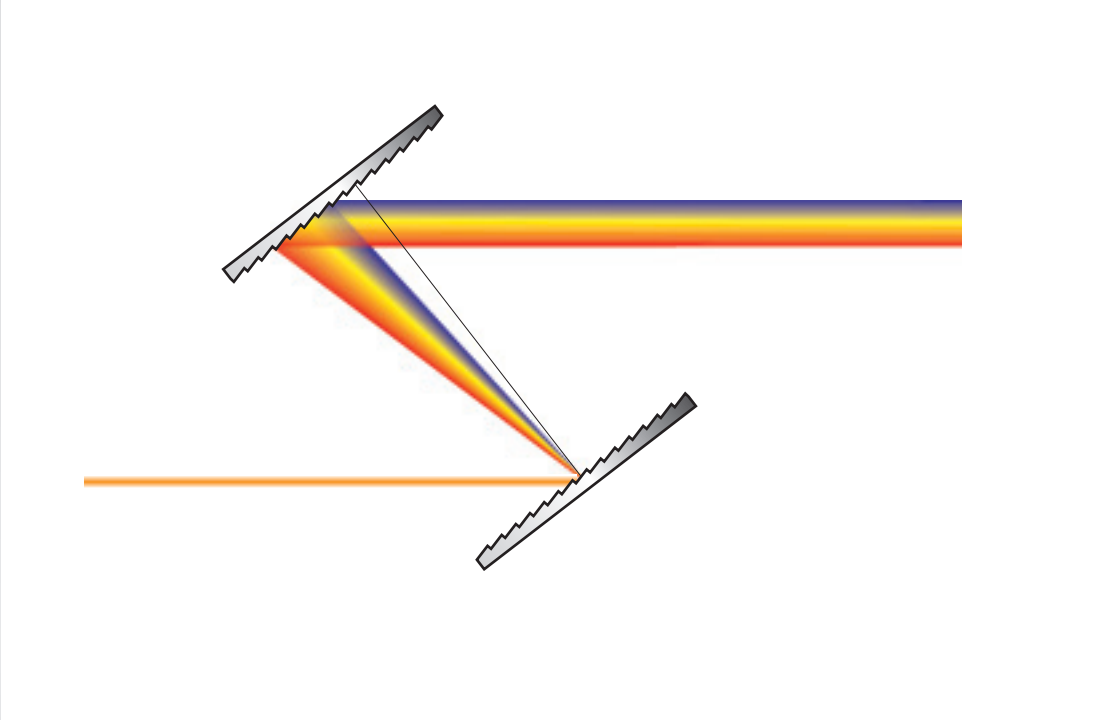
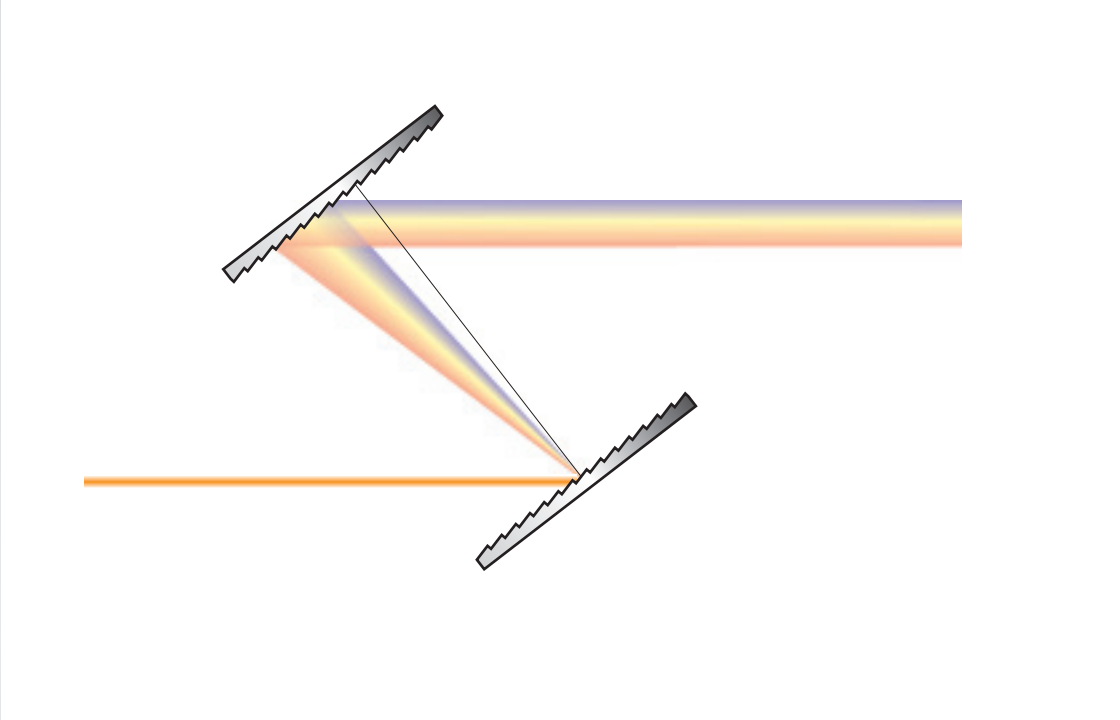


Pulse dispersion compensation

Does path length difference compensate?



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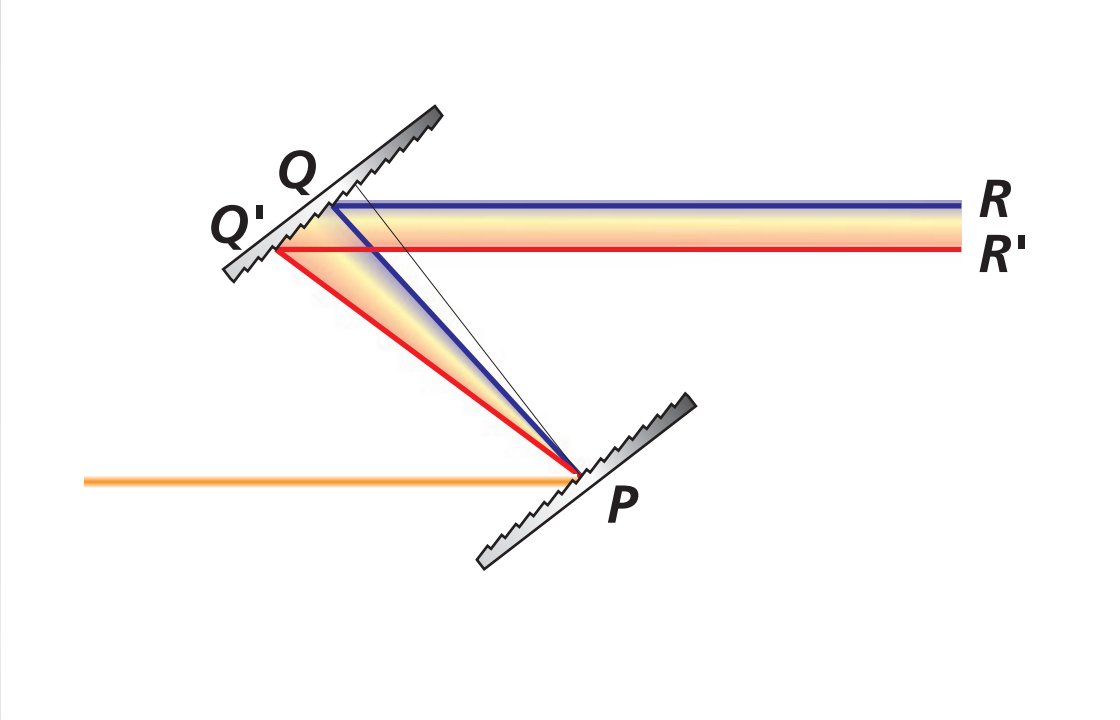

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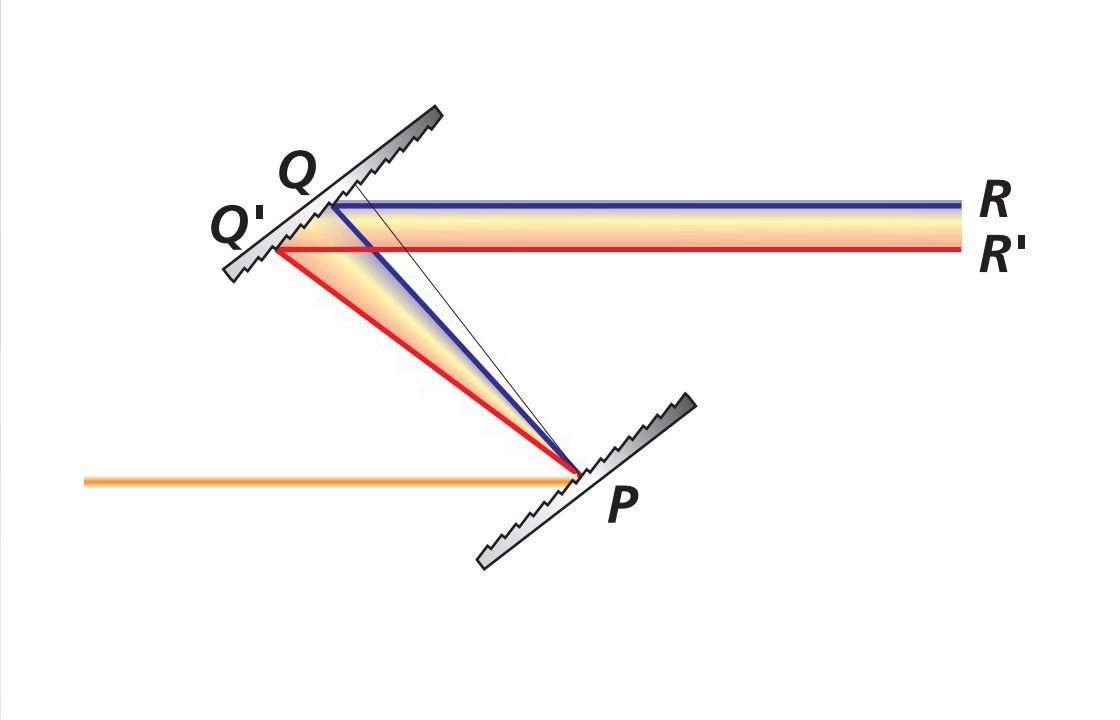
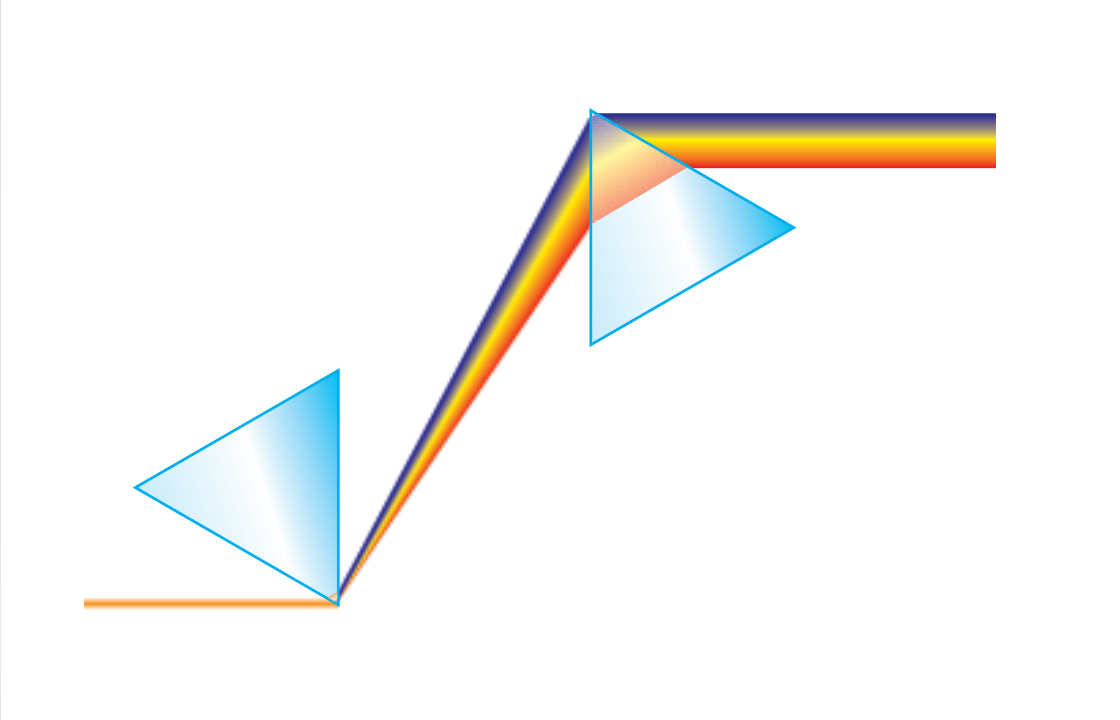


grating gives low frequency longer path length!

Pulse dispersion compensation

Does path length difference compensate?



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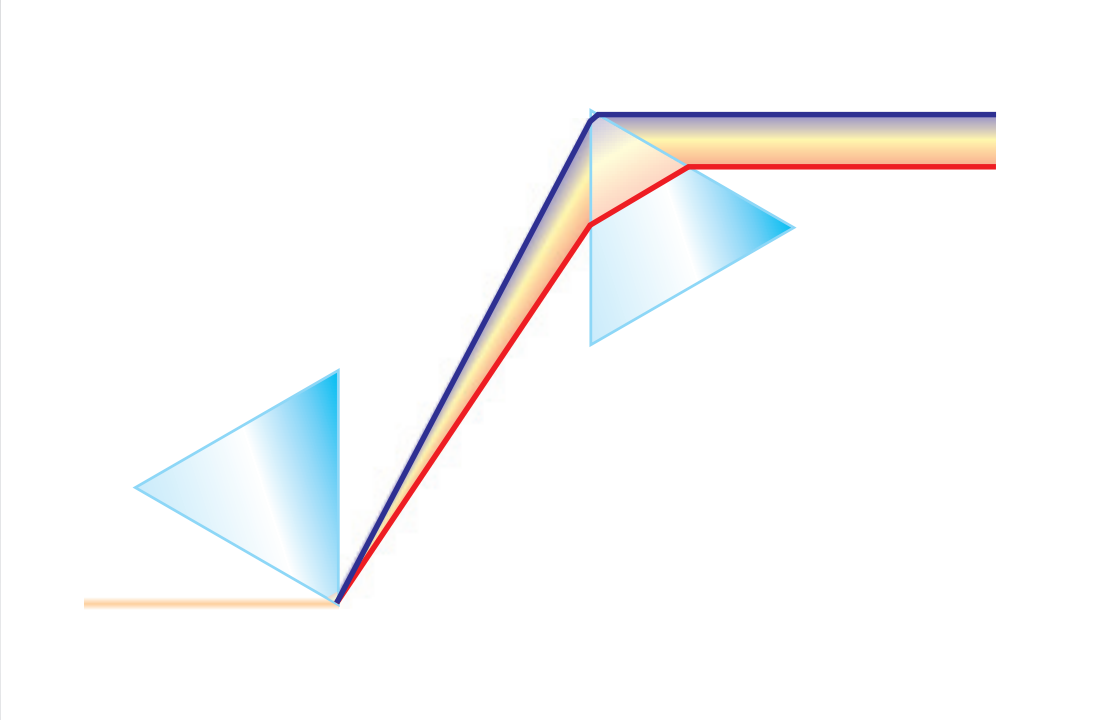

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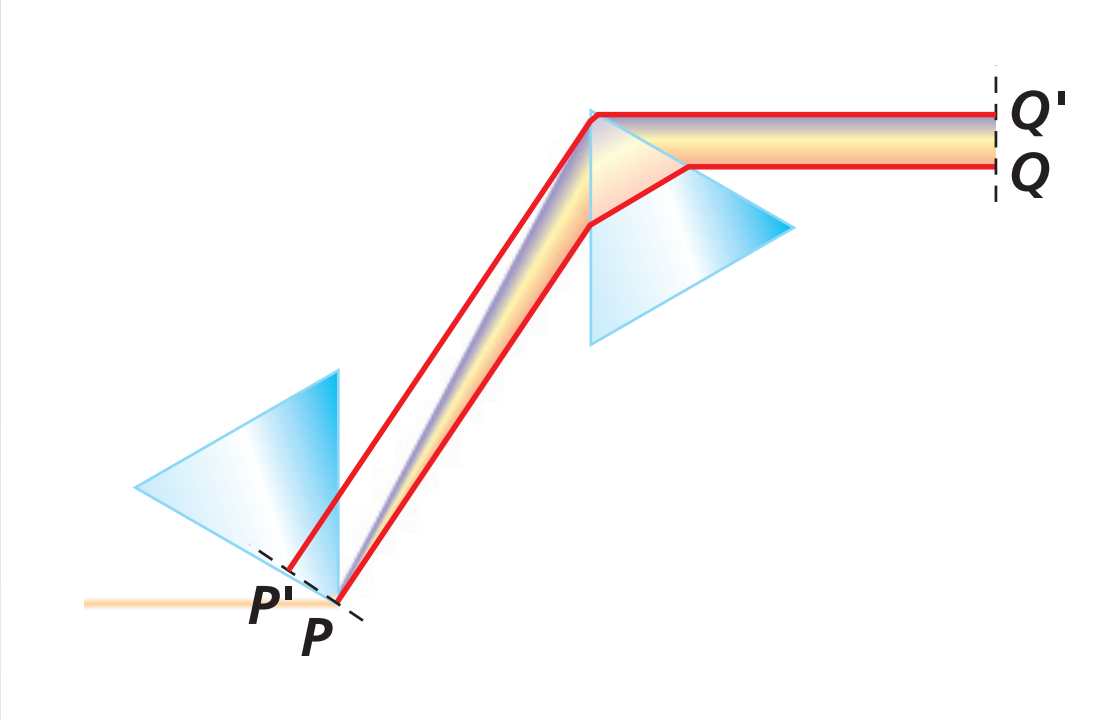

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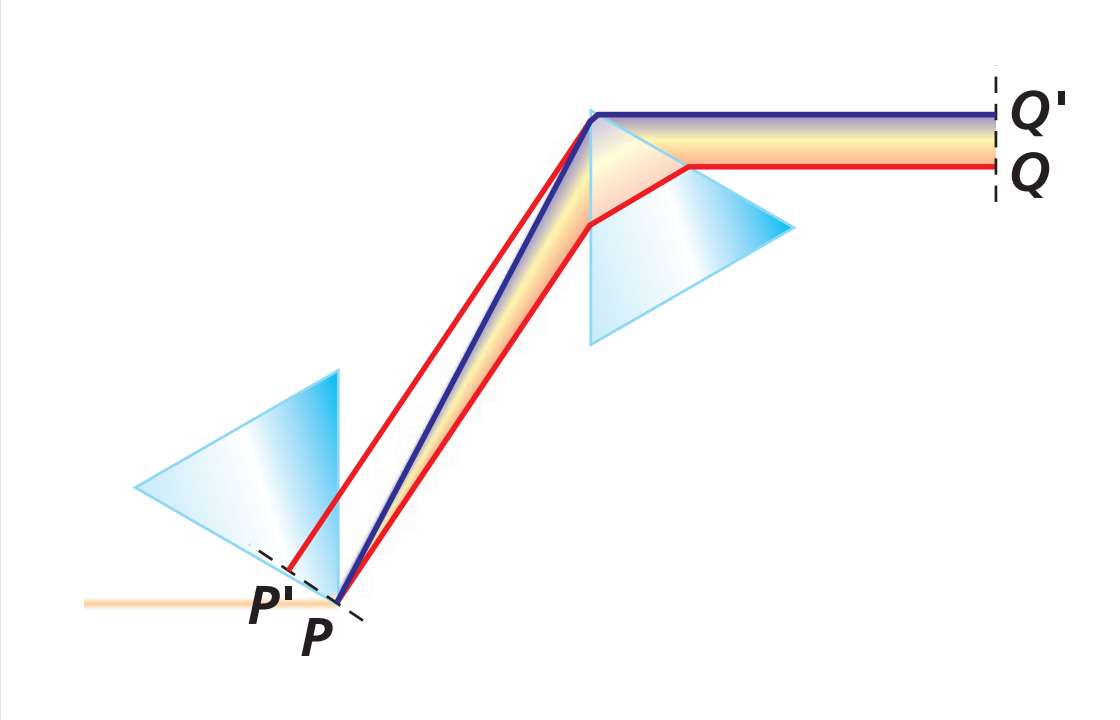

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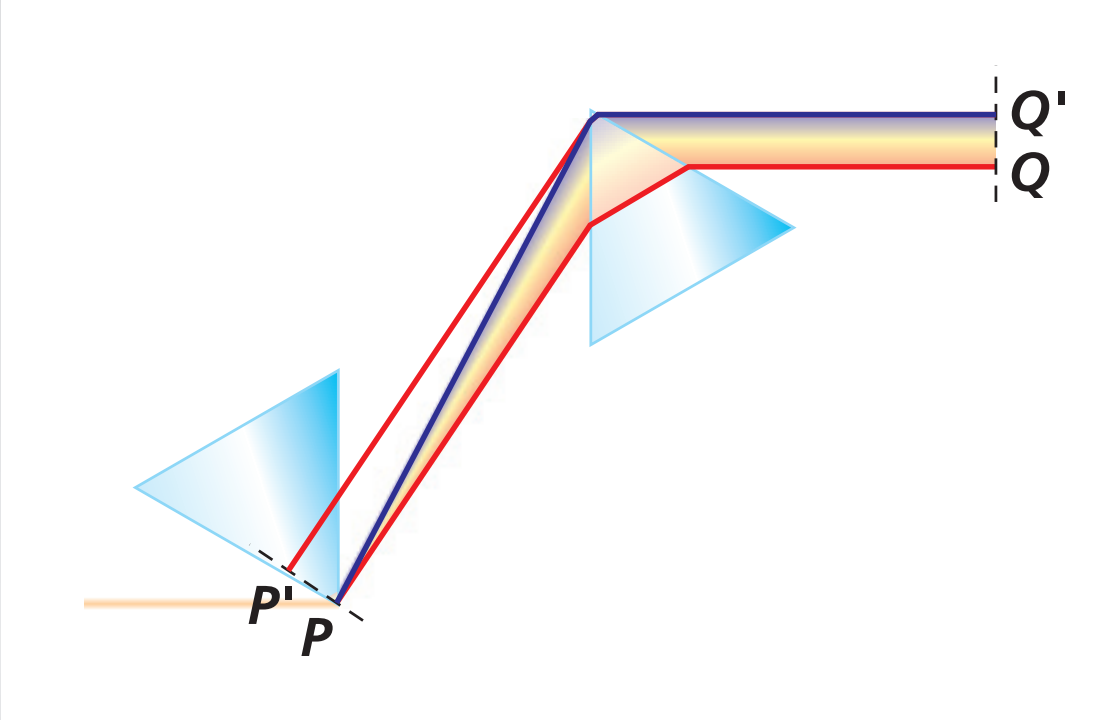

…so prism gives low frequency *shorter* path length!

Pulse dispersion compensation

Consider traveling Gaussian pulse:

$$y(t) = \exp\left[-\frac{(x - v_g t)^2}{2\sigma_t^2}\right]\sin(kx - \omega t)$$

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1. **Yes, it is dispersive**
2. **No, it is not dispersive (pulse shape is constant)**
3. **Cannot tell**

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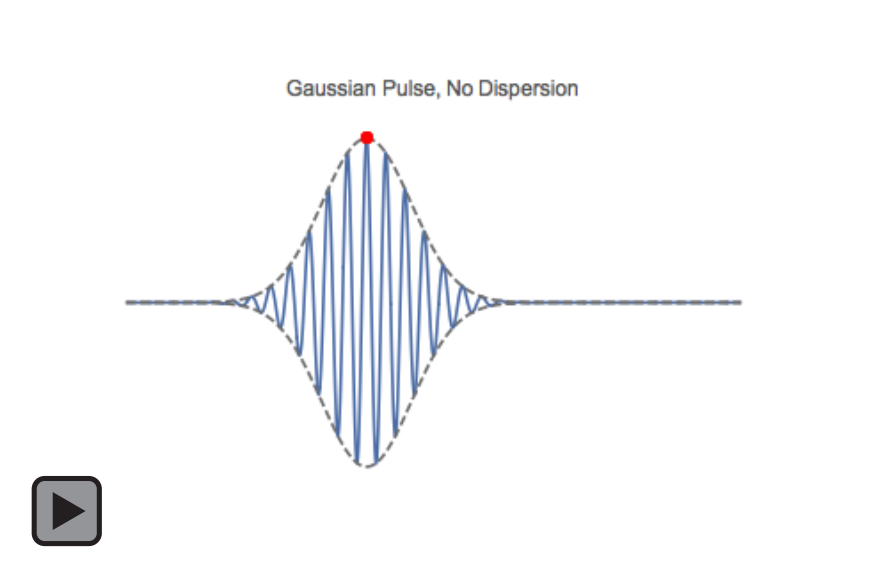
…but Gaussian shape of pulse is constant!

Pulse dispersion compensation

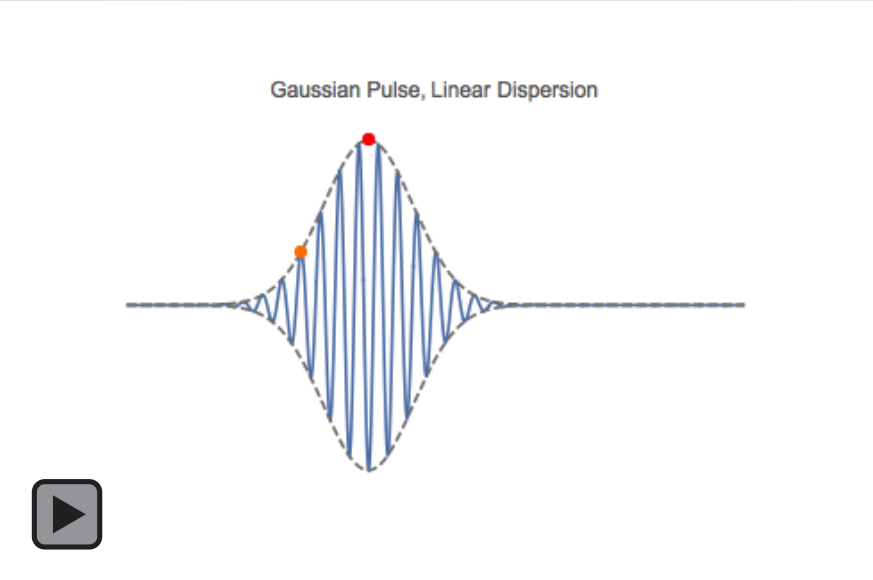
Gaussian, no dispersion



$$y(t) = \exp\left[-\frac{(x - v_g t)^2}{2\sigma_t^2}\right]\sin(kx - \omega t) \qquad v_g = \frac{\omega}{k}$$

Pulse dispersion compensation

Gaussian, dispersion



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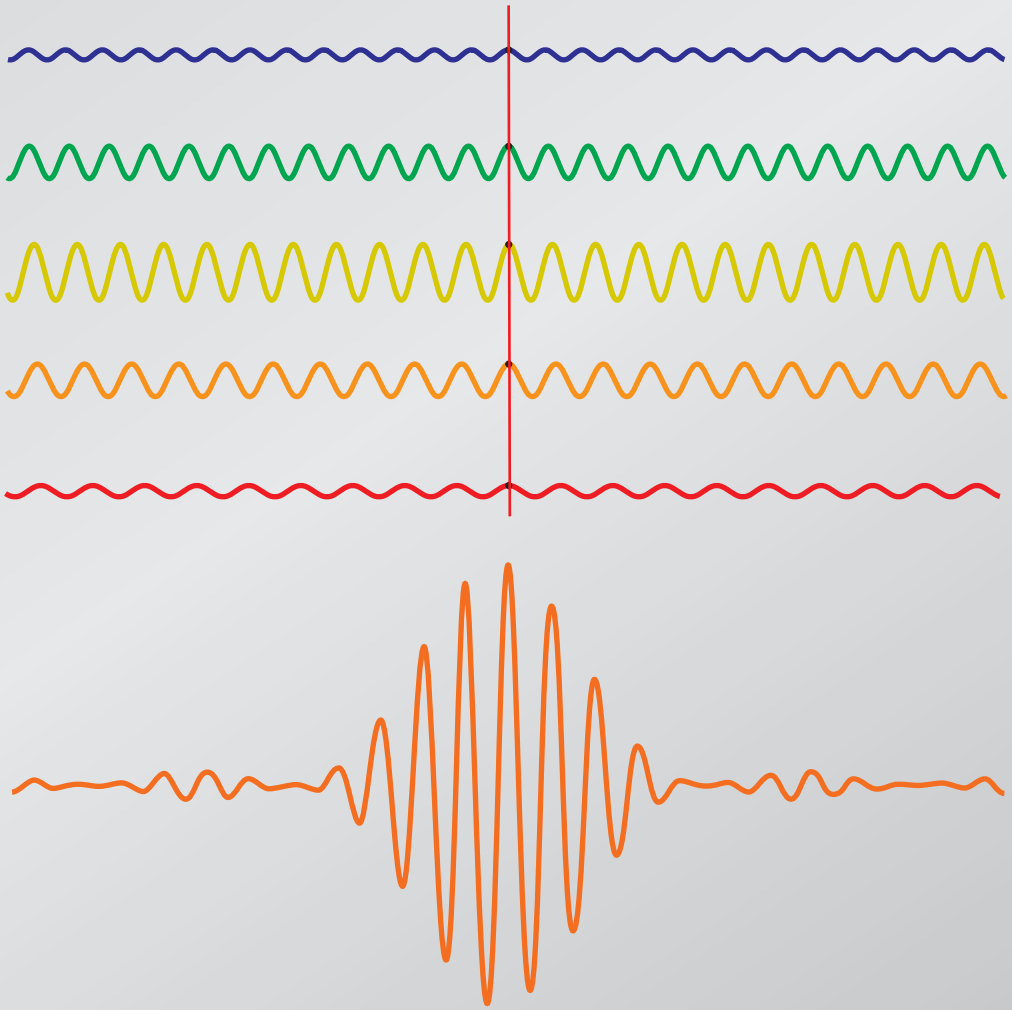
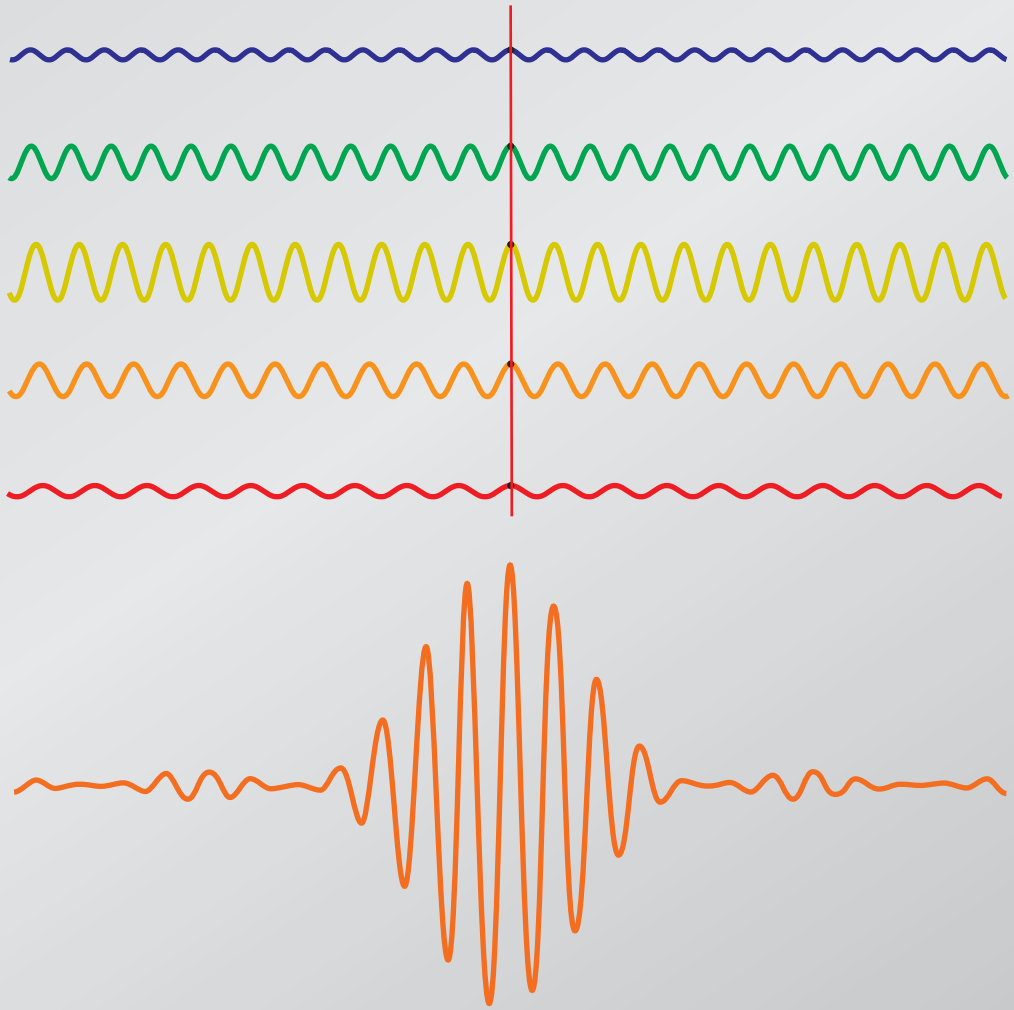
Pulse dispersion compensation



Pulse dispersion compensation

linear dispersion



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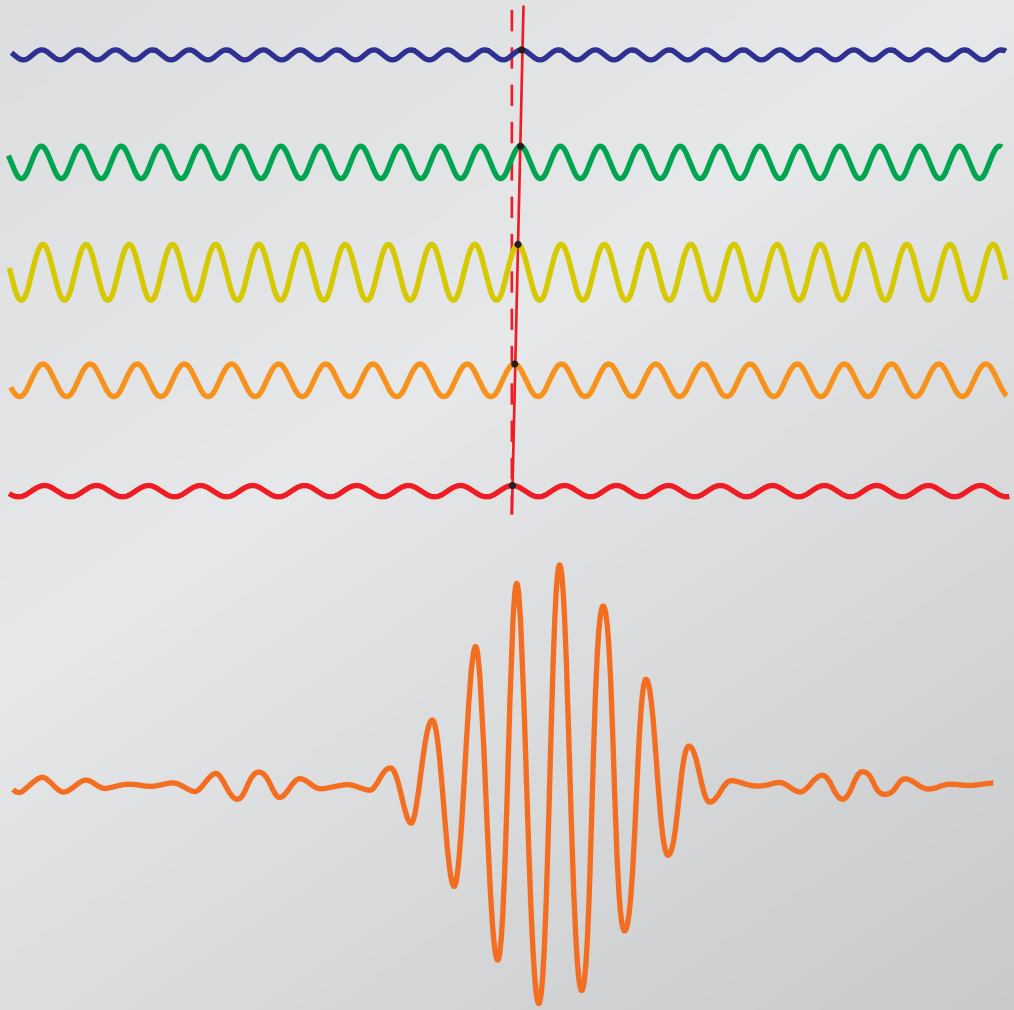

Pulse dispersion compensation

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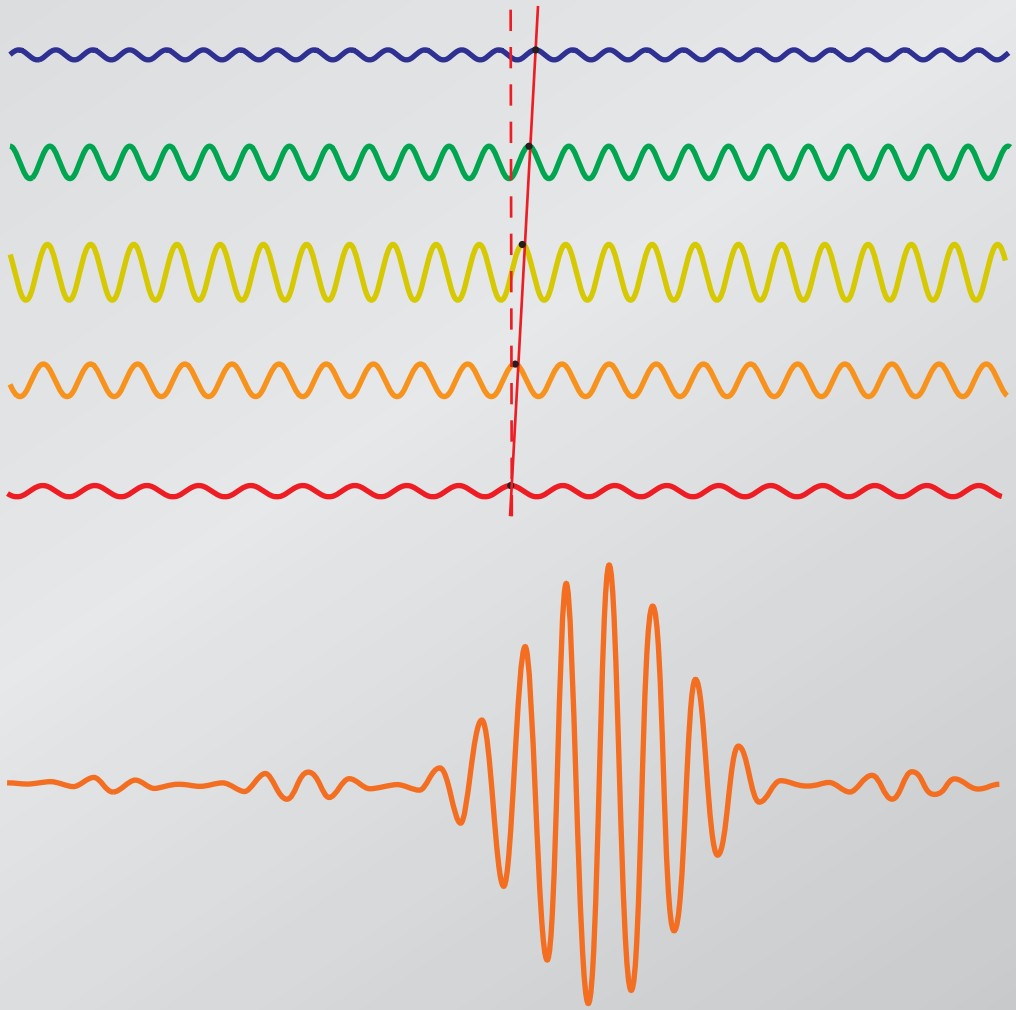

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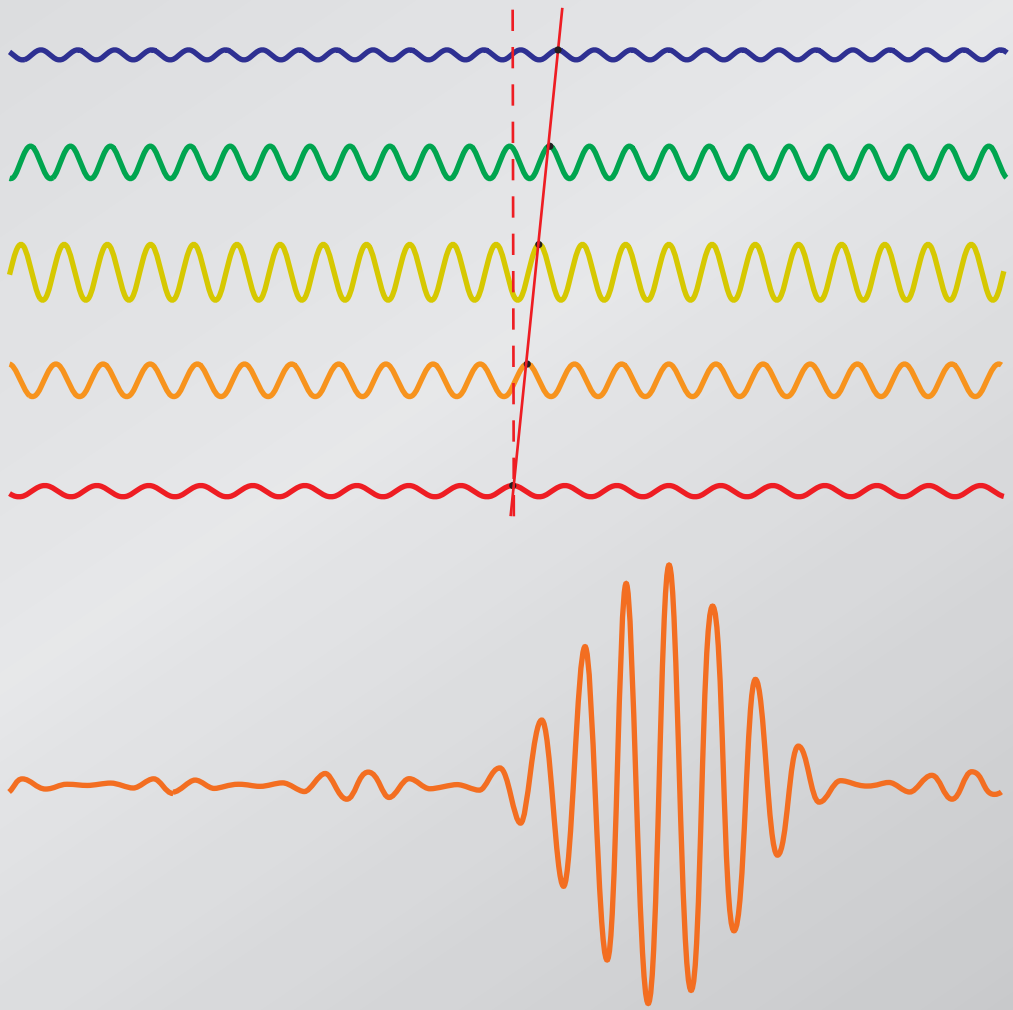

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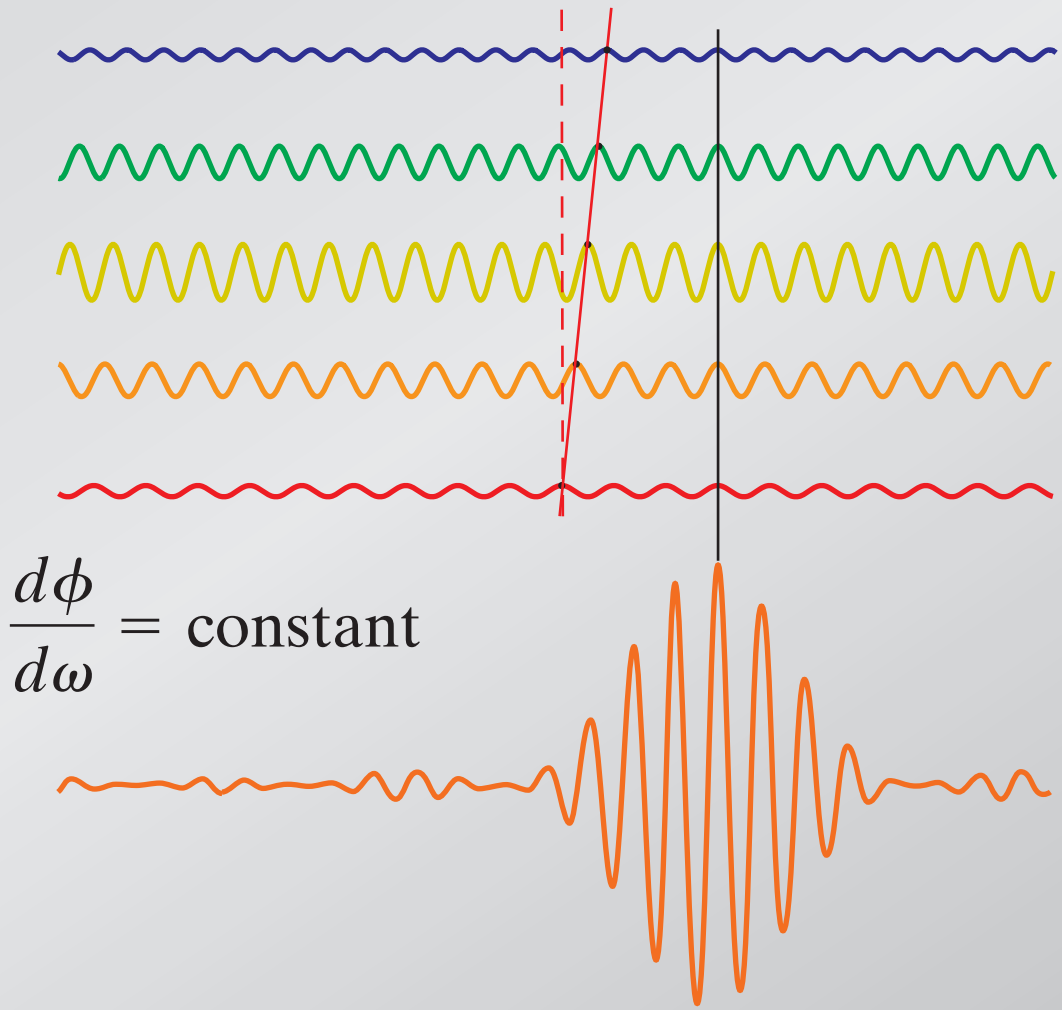
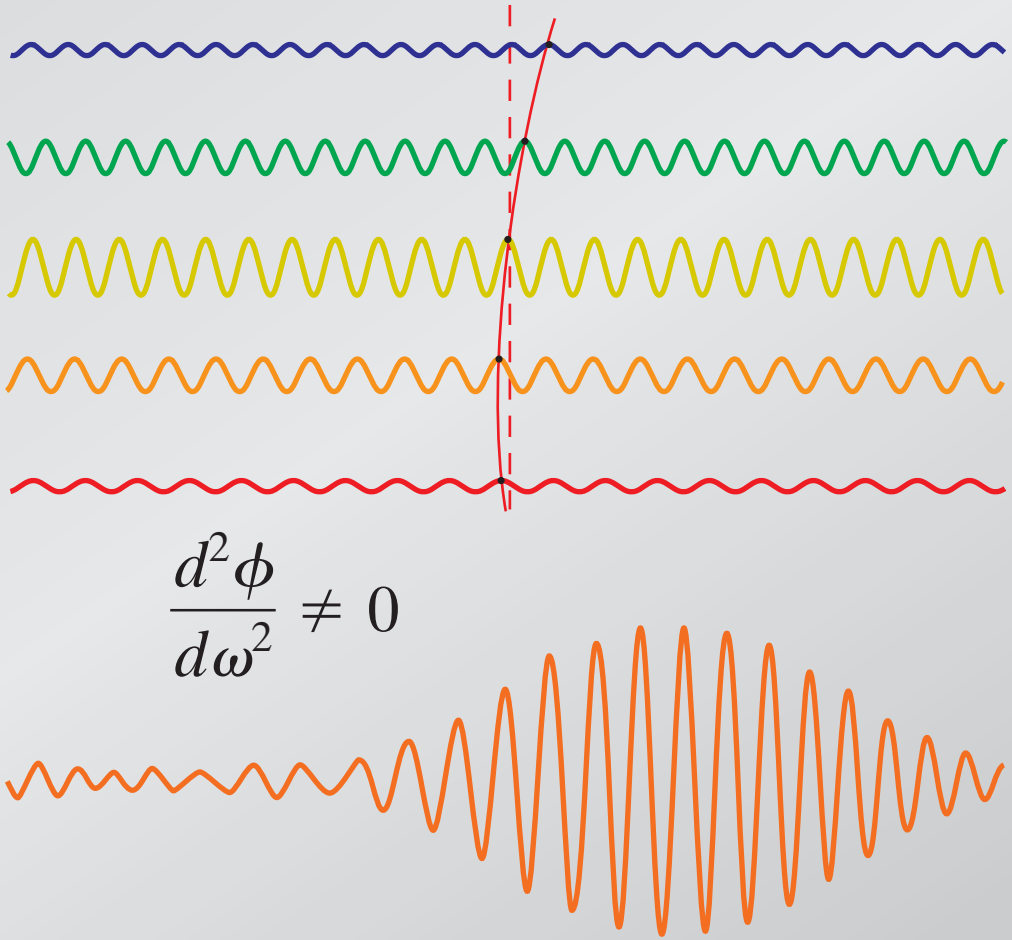


$$\frac{d\phi}{d\omega} = \text{constant}$$

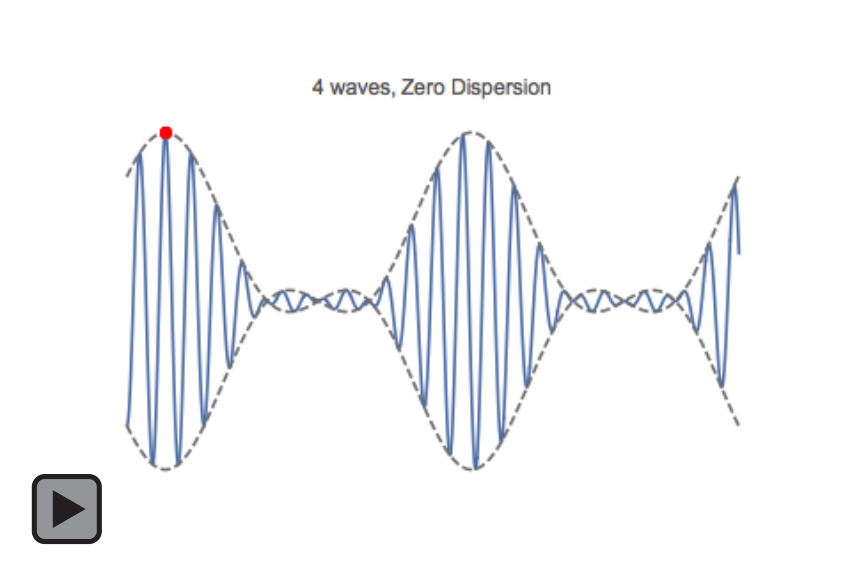
Pulse dispersion compensation

only *nonlinear* dispersion changes pulse shape!



Pulse dispersion compensation

4 sine waves, no dispersion



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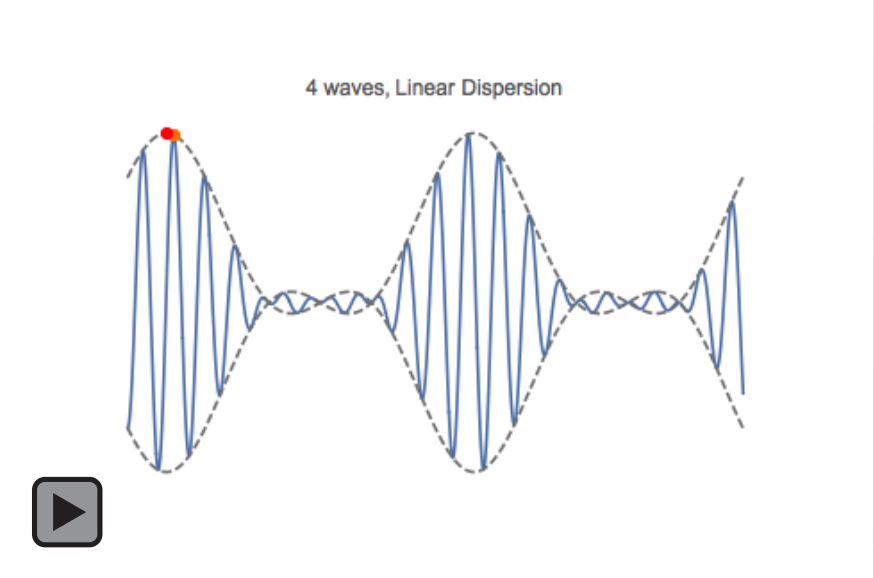

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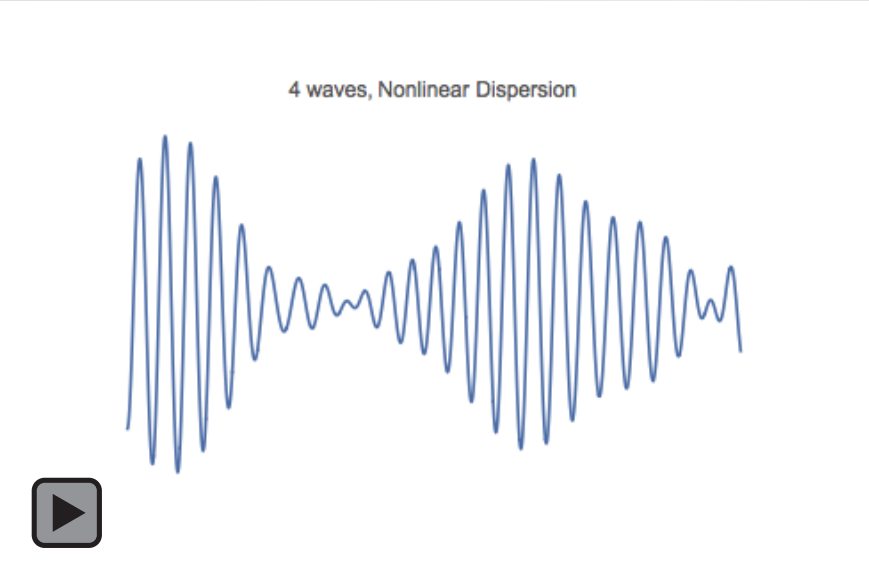

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Write dispersion as Taylor series:

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let

$$u \equiv \left(\frac{d\omega}{dk}\right)_{k=k_o} \quad \textbf{and} \quad w \equiv \left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}.$$

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if $w = 0$, then group velocity and pulse shape constant!

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Pulse dispersion compensation

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Pulse dispersion compensation

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dispersion	+	+
gratings	–	–
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Dispersion of pulses

Key points

- frequency-dependence of phase velocity distorts pulses
- group velocity describes motion of envelope
- group and phase velocity can be positive or negative
- group and phase velocity can exceed c

Outline

- optical properties of materials
- dispersion of pulses
- **nonlinear optics**
- waveguiding
- engineering the index

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